

# Strict inequalities of critical values in continuum percolation

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## Abstract

We consider the supercritical finite-range random connection model where the points  $x, y$  of a homogeneous planar Poisson process are connected with probability  $f(|y-x|)$  for a given  $f$ . Performing percolation on the resulting graph, we show that the critical probabilities for site and bond percolation satisfy the strict inequality  $p_c^{\text{site}} > p_c^{\text{bond}}$ . We also show that reducing the connection function  $f$  strictly increases the critical Poisson intensity.

Finally, we deduce that performing a spreading transformation on  $f$  (thereby allowing connections over greater distances but with lower probabilities, leaving average degrees unchanged) *strictly* reduces the critical Poisson intensity. This is of practical relevance, indicating that in many real networks it is in principle possible to exploit the presence of spread-out, long range connections, to achieve connectivity at a strictly lower density value.

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## 1 Introduction

Since exact formulae for critical values in percolation are known only for a few special cases, it is of interest to obtain partial information in the form of inequalities between critical values for different percolation models. This is especially true for continuum percolation; no exact critical values at all are known in the continuum, while on the other hand some interesting inequalities have been discovered. A striking result of this type says that for percolation of copies of a fixed convex shape of unit area centred at Poisson points in the plane, the critical intensity is less for a triangle than for any other shape. This was established as a weak inequality by Jonasson (2001), and as a strict inequality by Roy and Tanemura (2002).

The present paper is concerned with another result of this type, which says that for the random connection model over Poisson points in the plane (or in higher dimensions), the critical intensity is decreased under a spreading transformation of the connection function, whereby connections between more distant points are allowed but with lower probability, so that the average degree remains unchanged.

A related topic is the comparison of critical values for bond and site percolation. Given an infinite connected graph  $G$ , let us denote these critical values by  $p_c^{\text{bond}}$  and  $p_c^{\text{site}}$ , respectively. The weak inequality  $p_c^{\text{site}} \geq p_c^{\text{bond}}$  can easily be proven by dynamic coupling, see for example Chapter 2 of Franceschetti and Meester (2007). If  $G$  is a rooted tree, then it is easy to see that  $p_c^{\text{site}} = p_c^{\text{bond}}$ , as each vertex, other than the root, can be uniquely identified by an edge and vice versa. By adding finitely many edges to an infinite tree, one can also construct other connected graphs for which the equality holds. On the other hand, the strict inequality  $p_c^{\text{site}} > p_c^{\text{bond}}$  has also been shown to hold in many circumstances. Grimmett and Stacey (1998) proved it for a large class of ‘finitely transitive’ graphs including the  $d$ -dimensional hypercubic lattices.

In this paper we show  $p_c^{\text{site}} > p_c^{\text{bond}}$  for certain *random* graphs arising in *continuum* percolation. Such graphs are not covered by previous results

because they are not finitely transitive; since their node degrees are not bounded, the group action defined by their automorphisms has infinitely many orbits. Continuum percolation graphs are of particular interest in the context of communication networks and are treated extensively in the books by Franceschetti and Meester (2007), Meester and Roy (1996), and Penrose (2003).

We consider the *random connection model* (RCM) of continuum percolation, which is defined as follows. Let  $\lambda > 0$  and let  $f : \mathbb{R}^+ \rightarrow [0, 1]$  (the so-called *connection function*) be specified. Let  $\mathcal{P}_\lambda$  be a homogeneous Poisson point process in the plane of intensity  $\lambda$  and connect every pair of points  $x, y \in \mathcal{P}_\lambda$  with probability  $f(|x - y|)$ , where  $|\cdot|$  denotes the Euclidean norm (*Gilbert's graph* is the special case of the RCM with  $f \equiv \mathbf{1}_{[0,1]}$ ). Provided  $\lambda$  exceeds a critical value  $\lambda_f$  which depends on  $f$ , the RCM graph has an infinite component almost surely.

In Theorems 2.1 and 2.2, we prove that  $p_c^{\text{site}} > p_c^{\text{bond}}$  for the random graphs arising from the supercritical RCM using any nonincreasing connection function with finite range (including Gilbert's graph). In Theorem 2.3 we show that replacing the connection function  $f(\cdot)$  by a smaller connection function  $g(\cdot)$  causes the critical intensity to strictly increase, i.e.  $\lambda_g > \lambda_f$ . In Theorem 2.4 we consider the spreading transformation already mentioned, which is defined as follows. Given connection function  $f$  and given  $0 < p < 1$ , define the spread-out connection function  $S_p f$  by

$$S_p f(r) = p \cdot f(\sqrt{pr}). \quad (1.1)$$

Thus, the probabilities are reduced by a factor  $p$  but the function is spatially stretched so as to maintain the same expected number of connections per node; see Figure 1 for a visual representation.

Theorem 2.4 says that for any nonincreasing connection function  $f$  with finite range, and for  $0 < p < 1$ , we have  $\lambda_{S_p f} < \lambda_f$  (the weak version of this inequality is much simpler, see Franceschetti et al. (2005)). Consequently, as  $p \downarrow 0$  the approach of the critical value  $\lambda_{S_p f}$  to its limiting value (which is known to equal  $(2\pi \int_0^\infty r f(r) dr)^{-1}$ , see Penrose (1993)) is strictly monotone.

In applications this is of interest, as it shows that unreliable, spread-out connections are strictly advantageous for reaching connectivity at a given node density value. In communication networks, for example, the 'quality' of a communication link decreases as the distance between transmitter and receiver increases. Hence, connections can be established between nearby

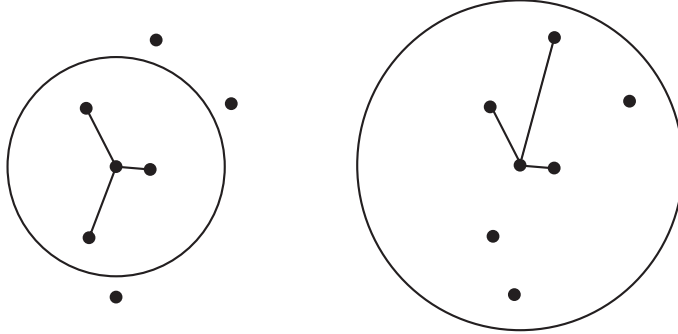


Figure 1: Left-hand side: RCM with connection function  $f(r)$ . Right-hand side: RCM with connection function  $p \cdot f(\sqrt{p}r)$ . Only connections to the centre point are depicted.

nodes, but reliable long-range connections are more difficult to obtain. Our results show that highly reliable, but short-range connections could be exchanged with less reliable, but longer-range connections, to obtain network connectivity more easily (in a strict sense), provided that the average number of functioning connections per node remains the same.

Our results carry over to a large class of connection functions having *infinite* range, i.e. with unbounded support. However, the proofs for this case require significant modifications and lengthy further arguments, and to keep the length of the current paper under control, we shall deal with the infinite range case elsewhere.

We conclude this section with some observations leading to some open problems. First, there exist spreading transformations for which the percolation threshold is unaffected. For example, an affine transformation of the plane converting discs into aligned ellipses of the same area would spread-out connection lengths but would not affect the percolation threshold. Second, there are spreading transformations for which neither weak nor strong inequalities are known. For example, the effect of the shifting and squeezing transformation for annuli considered by Franceschetti et al. (2005) and independently by Balister, Bollobás, and Walters (2004) is known only in the spread-out limit. Similar limiting results as the dimension of the space spreads to infinity are given by Meester, Penrose, and Sarkar (1997).

We are not aware of any lattice analogue of Theorem 2.4. In a lattice version of the spreading transformation, where one considers e.g. bond per-

colation on the vertices of  $\mathbb{Z}^d$  with range  $r$  becoming large, the critical value is known to approach its branching process limit (see Penrose (1993) or Bollobás, Janson, and Riordan (2005)), but the convergence is not known to be monotone.

## 2 Statement and discussion of results

For  $\lambda > 0$ , let  $\mathcal{P}_\lambda$  be a homogeneous Poisson process in  $\mathbb{R}^2$  of intensity  $\lambda$ . The random connection model (RCM) driven by  $\mathcal{P}_\lambda$ , with nonincreasing connection function  $f : \mathbb{R}^+ \rightarrow [0, 1]$ , is obtained by connecting each pair of points  $x, y \in \mathcal{P}_\lambda$  by an undirected edge with probability  $f(|x - y|)$ , independently of other pairs. We denote the resulting graph by  $RCM(\lambda, f)$ . For a formal description of the RCM, see Meester and Roy (1996). It is well known that provided  $0 < \int_0^\infty r f(r) dr < \infty$ , the RCM has a critical density value  $\lambda_f \in (0, \infty)$ , in fact with

$$\lambda_f \geq \frac{1}{2\pi \int_0^\infty r f(r) dr}, \quad (2.1)$$

such that if  $\lambda > \lambda_f$  then there exists a.s. a unique infinite connected component in  $RCM(\lambda, f)$ , while if  $\lambda < \lambda_f$  then there is a.s. no infinite connected component in  $RCM(\lambda, f)$ , see Penrose (1991), Meester and Roy (1996). When it exists, we denote this infinite component by  $\mathcal{C}$ .

In the site percolation model on  $\mathcal{C}$ , each vertex is independently marked open with probability  $p$ , and closed otherwise, and we look for an unbounded connected component in the subgraph  $\mathcal{C}_v$  induced by the open vertices. It is easy to see that this is equivalent to rescaling the original Poisson process to one with intensity  $p\lambda$  and looking for an unbounded connected component there. It follows that for  $\lambda > \lambda_f$  there is a critical value  $p_c^{\text{site}} \in (0, 1)$  (namely  $p_c^{\text{site}} = \lambda_f/\lambda$ ) such that if  $p > p_c^{\text{site}}$  then there is a.s. an infinite connected component in  $\mathcal{C}_v$ , and if  $p < p_c^{\text{site}}$  then there is a.s. no such infinite component.

In the bond percolation model on  $\mathcal{C}$ , we independently declare each edge to be open with probability  $p$ , and closed otherwise, and look for an unbounded connected component in the subgraph  $\mathcal{C}_e$  induced by the open edges. This is equivalent to constructing an RCM with the original connection function  $f$  replaced by  $p \cdot f$ . There is a critical probability  $p_c^{\text{bond}} \in (0, 1)$  such that if  $p > p_c^{\text{bond}}$  then there is a.s. an infinite connected component in  $\mathcal{C}_e$ , and if  $p < p_c^{\text{bond}}$  then there is a.s. no such infinite component. To verify

that  $p_c^{\text{bond}} \in (0, 1)$ , observe that  $p_c^{\text{bond}} \leq p_c^{\text{site}} < 1$  while by (2.1) we have  $p_c^{\text{bond}} \geq (2\pi\lambda \int_0^\infty r f(r) dr)^{-1}$ .

A special case of the RCM arises when  $f(r) = 1$  for  $r \leq 1$  and  $f(r) = 0$  for  $r > 1$ . This is called Gilbert's graph,  $G(\mathcal{P}_\lambda, 1)$ , and is formed from  $\mathcal{P}_\lambda$  by joining every two points  $x, y \in \mathcal{P}_\lambda$  with  $|x - y| \leq 1$ . We denote the critical value of  $\lambda$  by  $\lambda_c$  in this case.

Our first results provide strict inequalities between  $p_c^{\text{site}}$  and  $p_c^{\text{bond}}$ . We first prove our results for Gilbert's graph (Theorem 2.1) and then generalise to the random connection model; the proof for the first case sets up many of the arguments in the more general case.

**Theorem 2.1** *Consider  $G(\mathcal{P}_\lambda, 1)$  for  $\lambda > \lambda_c$ . On  $\mathcal{C}$  we have  $p_c^{\text{site}} > p_c^{\text{bond}}$ .*

The next result generalizes Theorem 2.1, and concerns the RCM with connection function  $f$  having bounded support.

**Theorem 2.2** *Consider  $\text{RCM}(\lambda, f)$  for  $\lambda > \lambda_f$ . If  $f$  is nonincreasing and  $0 < \sup\{a : f(a) > 0\} < \infty$  then on  $\mathcal{C}$  we have  $p_c^{\text{site}} > p_c^{\text{bond}}$ .*

Our next result provides a strict inequality governing the effect on the RCM critical intensity  $\lambda_f$  if one reduces the connection function ( $f(r), r \geq 0$ ) by a constant factor, i.e. if one uses instead the 'squashed' connection function  $pf := (pf(r), r \geq 0)$  with  $p \in (0, 1)$  a constant. The weak inequality  $\lambda_{pf} \geq \lambda_f$  is clear, and the next result improves it to a strict inequality.

**Theorem 2.3** *Let  $q_0 \in (0, 1)$ . Suppose the connection function  $f$  is nonincreasing and  $0 < \sup\{a : f(a) > 0\} < \infty$ . Then  $\lambda_{q_0 f} > \lambda_f$ .*

In fact, Theorem 2.3 holds in greater generality: if  $f$  and  $g$  are connection functions, satisfying the hypotheses of Theorem 2.2, with  $g(r) \leq f(r)$  for all  $r$  and  $g(r) < f(r)$  for  $r$  in some sub-interval of  $(0, \infty)$ , then  $\lambda_g < \lambda_f$ . This can be proved by an extension of the proof of Theorem 2.3, which we omit.

Our last result is concerned with the spreading-out transformation  $S_p$  defined by (1.1).

**Theorem 2.4** *Suppose  $0 < p < q \leq 1$ . Suppose the connection function  $f$  is nonincreasing and  $0 < \sup\{a : f(a) > 0\} < \infty$ . Then  $\lambda_{S_p f} < \lambda_{S_q f}$ . Also, the inequality (2.1) is strict.*

To conclude this section we give an overview of the technique of proof and related literature. We note first that our proof of all of these results easily extends to 3 or more dimensions.

The basic strategy is to adapt the enhancement technique developed for percolation on lattices by Menshikov (1987), Aizenman and Grimmett (1991), Grimmett and Stacey (1998). This consists of constructing an ‘enhanced’ version of the site percolation process (i.e., one with some extra open sites added according to certain rules), for which the critical probability is *strictly less* than that of the original site process. Then one can use dynamic coupling of the enhanced model with bond percolation to complete the proof.

We face two main difficulties when trying to extend the enhancement technique to a continuum random setting. One of these amounts to constructing the desired enhancement on a random graph rather than on a deterministic one. The second one consists in adapting some basic inequalities for the enhanced graph, given in the discrete setting by Aizenman and Grimmett (1991), to the continuum setting. Because the possible configurations outside a given region now provide a continuum of possible boundary conditions, this requires somehow more involved geometric constructions and a careful incremental build-up of the Poisson point process. Once we circumvent these obstacles, it is not too difficult to obtain the final result using a classic dynamic coupling construction.

In order to keep the main ideas of the proof clear, we first prove Theorem 2.1, and later adapt the proof to the general case of Theorem 2.2. The proof of Theorem 2.3 uses an argument involving ‘diminishment’, rather than enhancement, of site percolation, and the proof of Theorem 2.4 uses the preceding results along with a coupling argument related to that used by Franceschetti et al. (2005) to get the weak version of Theorem 2.4.

The enhancement strategy has proven useful to show strict inequalities in a variety of contexts: Bezuidenhout, Grimmett, and Kesten (1993), and Grimmett (1994), use this technique in the context of Potts and random cluster models; Roy, Sarkar, and White (1998) use it in the context of directed percolation. In the continuum, Sarkar (1997) uses enhancement to demonstrate coexistence of occupied and vacant phases for the three-dimensional Poisson Boolean model. Roy and Tanemura (2002) use it in the context of percolation of different convex shapes.

### 3 Gilbert's Graph: Proof of Theorem 2.1

We now describe the enhancement needed to prove Theorem 1. Throughout this section we consider Gilbert's graph  $G(\mathcal{P}_\lambda, 1)$  with  $\lambda > \lambda_c$ . The objective is to describe a way to add open vertices to the site percolation model without changing the coupled bond percolation model. To do so, we introduce two kinds of coloured vertices, red vertices (the original open vertices) and green vertices (closed vertices which have been enhanced) and for any two vertices  $x, y$  we write that  $x \sim y$  if they are joined by an edge. In  $G(\mathcal{P}_\lambda, 1)$ , if we have vertices  $v, w, x, y, z$  such that  $x$  is closed, has no neighbours other than  $v, w, y, z$ , which are all red, and  $v \sim w$  and  $y \sim z$  but there are no other edges amongst  $v, w, y$  and  $z$  then we say  $x$  is *correctly configured* in  $G(\mathcal{P}_\lambda, 1)$ , and refer to this as a *bow tie* configuration of edges. If a vertex  $x$  is correctly configured we make it green with probability  $q$ , independently of everything else; see Figure 2.

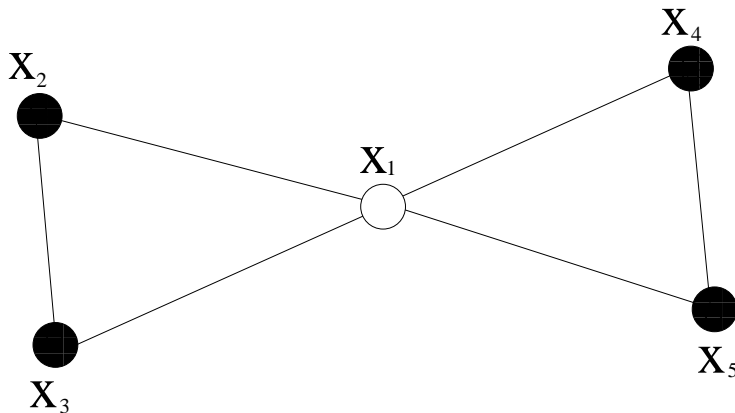


Figure 2: The bow tie enhancement.

Let  $B_n$  be the open disc of radius  $n$  centred at the origin. Let  $\underline{Y} = (Y_i, i \geq 0)$  and  $\underline{Z} = (Z_i, i \geq 0)$  be sequences of independent uniform  $[0, 1]$  random variables. List the vertices of  $\mathcal{P}_\lambda$  in order of increasing distance from the origin as  $x_1, x_2, x_3, \dots$ . Declare a vertex  $x_i$  to be *red* if  $Y_i < p$  and *closed* otherwise. Once the sets of red and closed vertices have been decided in this way, apply the enhancement by declaring each closed vertex  $x_j$  to be *green* if it is correctly configured and  $Z_j < q$ . We shall sometimes need to consider

the Poisson process with an extra vertex inserted at  $x \in B_n$ , in which case the extra vertex has values  $Y_0$  and  $Z_0$  associated with it. We shall refer to vertices that are either red or green as being *coloured*.

Let  $\partial B_n$  be the annulus  $B_n \setminus B_{n-0.5}$  and let  $A_n$  be the event that for the Poisson process  $\mathcal{P}_\lambda \cap B_n$ , there is a path from a coloured vertex in  $B_{0.5}$  to a coloured vertex in  $\partial B_n$  in  $G(\mathcal{P}_\lambda, 1) \cap B_n$  using only coloured vertices (note that  $A_n$  is based on a process completely inside  $B_n$ ; we do not allow vertices outside of  $B_n$  to affect possible enhancements inside  $B_n$ ). For  $x \in B_n$ , let  $A_n^x$  be defined the same way as  $A_n$ , but in terms of the point process  $(\mathcal{P}_\lambda \cap B_n) \cup \{x\}$ , i.e. the Poisson process in  $B_n$  with a point inserted at  $x$ .

Let  $\theta_n(p, q)$  be the probability that  $A_n$  occurs, and define

$$\theta(p, q) \equiv \liminf_{n \rightarrow \infty} (\theta_n(p, q)).$$

The following proposition states that  $\theta(p, q)$  is indeed the percolation function associated to the enhanced model. From now on we use ‘vertex’ to refer to a point of the Poisson process and ‘point’ to refer to an arbitrary location in  $\mathbb{R}^2$ .

**Proposition 3.1** *There is a.s. an infinite connected component in  $G(\mathcal{P}_\lambda, 1)$  using only red and green vertices if and only if  $\theta(p, q) > 0$ .*

**Proof of Proposition 3.1.** For the if part let  $A'_n$  be the event that there is a coloured path from  $B_{0.5}$  to outside  $B_{n-2}$ , so  $A_n$  is contained in  $A'_n$ . Let  $\phi_n(p, q)$  be the probability of  $A'_n$  occurring (which is monotone in  $n$ ), and let  $\phi(p, q)$  be the limit as  $n$  goes to  $\infty$ . Therefore  $\phi_n(p, q) \geq \theta_n(p, q)$  for all  $n$  so  $\phi(p, q) \geq \theta(p, q) > 0$ , but  $\phi(p, q)$  is just the probability of there being an infinite coloured component intersecting  $B_{0.5}$  and it is well known that there is almost surely an infinite coloured component if  $\phi(p, q) > 0$ .

For the only if part, if there is almost surely an infinite component then  $\phi(p, q) > 0$ . Given  $n \geq 6$ , we build up the Poisson process on the whole of  $B_{n-3}$ . If there are any closed vertices that are not definitely correctly or incorrectly configured, we build up the process in the rest of their 1-neighbourhood, and this determines whether they are green or uncoloured. If any more closed vertices occur they cannot be correctly configured as they will be joined to a closed vertex. Therefore we have built up the process everywhere in a region  $R$  with  $B_{n-3} \subset R \subset B_{n-2}$ , and all uncoloured vertices at this stage will remain uncoloured. Let  $V$  be the set of coloured vertices that are joined by a coloured path to a coloured vertex in  $B_{0.5}$  at this stage.

Next, we build out the process radially symmetrically from  $B_{n-3}$  (apart from where the process has already been built up) until a vertex  $v$  occurs that is connected to a vertex in  $V$ . Let  $J$  be the event that such a vertex  $v$  occurs, so  $J$  must occur for  $A'_n$  to occur. Assuming  $J$  occurs, set  $r = |v|$ , so  $r \in [n-3, n-1)$ . Then we can find points  $a_1, a_2, \dots, a_9$  on the line  $0v$  extended away from the origin such that  $a_1$  is  $r+0.3$  from the origin,  $a_2$  is  $r+0.6$  from the origin and so on. Surround  $a_1, \dots, a_9$  with circles  $D_1, \dots, D_9$  of radius 0.05 around them. If there is at least one red vertex in each one of these little circles that is contained in  $B_n$  when the process continues to the whole of  $B_n$ , and  $v$  is also red then  $A_n$  occurs. Therefore if  $J$  occurs then the conditional probability of  $A_n$  occurring is at least  $\gamma$ , where

$$\gamma = p(1 - \exp(-0.0025\lambda p\pi))^9,$$

as this is the probability of getting at least one red vertex in each little circle and  $v$  being red. Therefore  $\theta_n(p, q) \geq \gamma P[J] \geq \gamma \phi(p, q)$  for all  $n \geq 6$ , so  $\theta(p, q) \geq \gamma \phi(p, q) > 0$ .  $\square$

Our next lemma provides an analogue of the Margulis-Russo formula for the enhanced continuum model. First, we need to introduce the notion of pivotal vertices.

Given the configuration  $(\mathcal{P}_\lambda, \underline{Y}, \underline{Z})$  and inserting a vertex at  $x$  we say that  $x$  is *1-pivotal* in  $B_n$  if putting  $Y_0 = 0$  means that  $A_n^x$  occurs but putting  $Y_0 = 1$  means it does not. Notice that  $x$  can either complete a path (but it cannot do via being enhanced), or it could make another closed vertex correctly configured which in turn would complete a path. We say that  $x$  is *2-pivotal* in  $B_n$  if inserting a vertex at  $x$  and putting  $Z_0 = 0$  means  $A_n^x$  occurs but putting  $Z_0 = 1$  means it does not. That is,  $Y_0 > p$  and adding a closed vertex  $v$  at  $x$  means  $v$  is correctly configured and enhancing it to a green vertex means  $A_n^x$  occurs but otherwise it does not.

For  $i = 1, 2$  let  $E_{n,i}(x)$  be the event that  $x$  is  $i$ -pivotal in  $B_n$ , and set  $P_{n,i}(x, p, q) := P[E_{n,i}(x)]$ .

**Lemma 3.1** *For all  $n > 0.5$  and  $p \in (0, 1)$  and  $q \in (0, 1)$  it is the case that*

$$\frac{\partial \theta_n(p, q)}{\partial p} = \int_{B_n} \lambda P_{n,1}(x, p, q) \, dx \quad (3.1)$$

and

$$\frac{\partial \theta_n(p, q)}{\partial q} = \int_{B_n} \lambda P_{n,2}(x, p, q) \, dx. \quad (3.2)$$

**Proof.** Let  $\mathcal{F}$  be the  $\sigma$ -algebra generated by the locations but not the colours of the vertices of  $\mathcal{P}_\lambda \cap B_n$ . Let  $N_1$  be the number of 1-pivotal vertices. Define  $\mathcal{F}$ -measurable random variables,  $X_{p,q}$  and  $Y_{p,q}$  as follows;  $X_{p,q}$  is the conditional probability that  $A_n$  occurs, and  $Y_{p,q}$  is the conditional expectation of  $N_1$ , given the configuration of  $\mathcal{P}_\lambda$ . By the standard version of the Margulis-Russo formula for an increasing event defined on a finite collection of Bernoulli variables (Russo (1981), Lemma 3),

$$\lim_{h \rightarrow 0} h^{-1}(X_{p+h,q} - X_{p,q}) = Y_{p,q}, \quad a.s.$$

Let  $M$  denote the total number of vertices of  $\mathcal{P}_\lambda$  in  $B_n$ . By the standard coupling of Bernoulli variables, and Boole's inequality,  $|X_{p+h,q} - X_{p,q}| \leq |h|M$  almost surely, and since  $M$  is integrable, dominated convergence yields

$$\frac{\partial \theta_n(p, q)}{\partial p} = \lim_{h \rightarrow 0} E[h^{-1}(X_{p+h,q} - X_{p,q})] = E[Y_{p,q}] = E[N_1], \quad (3.3)$$

and by a standard application of the Palm theory of Poisson processes (see e.g. Penrose (2003)), the right hand side of (3.3) equals the right hand side of (3.1). The proof of (3.2) is similar.  $\square$

The key step in proving Theorem 1 is given by the following result.

**Lemma 3.2** *There is a continuous function  $\delta : (0, 1)^2 \rightarrow (0, \infty)$  such that for all  $n > 100$ ,  $x \in B_n$  and  $(p, q) \in (0, 1)^2$ , we have*

$$P_{n,2}(x, p, q) \geq \delta(p, q)P_{n,1}(x, p, q). \quad (3.4)$$

Before proving this, we give a result saying that we can assume there are only red vertices inside an annulus of fixed size. For  $x \in \mathbb{R}^2$ , and  $0 \leq \alpha < \beta$ , let  $C_\alpha(x)$  be the closed circle (i.e., disc) of radius  $\alpha$  centred at  $x$ , and let  $A_{\alpha,\beta}(x)$  denote the annulus  $C_\beta(x) \setminus C_\alpha(x)$ . Given  $n$  and given  $x \in B_n$ , let  $R_n(x, \alpha, \beta)$  be the event that all vertices in  $A_{\alpha,\beta}(x) \cap B_n$  are red.

**Lemma 3.3** *Fix  $\alpha > 3$  and  $\beta > \alpha + 3$ . There exists a continuous function  $\delta_1 : (0, 1)^2 \rightarrow (0, \infty)$ , such that for all  $(p, q) \in (0, 1)^2$ , all  $n > \beta + 3$  and all  $x \in B_n$  with  $|x| < \alpha - 2$  or  $|x| > \beta + 2$ , we have*

$$P[E_{n,1}(x) \cap R_n(x, \alpha, \beta)] \geq \delta_1(p, q)P_{n,1}(x).$$

**Proof.** We shall consider a modified model, which is the same as the enhanced model but with enhancements suppressed for all those vertices lying in  $A_{\alpha-1,\beta+1} := A_{\alpha-1,\beta+1}(x)$ . Let  $E'_{n,1}(x)$  be the event that  $x$  is 1-pivotal in the modified model.

Returning to the original model, first create the Poisson process of intensity  $\lambda$  in  $B_n$ . Then for all the vertices in  $B_n \setminus A_{\alpha-1,\beta+1}$ , decide whether they are red or closed. Then, for all those vertices in  $B_n \cap A_{\alpha-1,\beta+1}$  with more than 4 neighbours, or with at least one closed neighbour outside  $A_{\alpha-1,\beta+1}$ , decide whether they are red or closed. This decides whether or not they are coloured as these vertices cannot possibly become green because they are not correctly configured. We now can tell which of the closed vertices outside  $A_{\alpha-1,\beta+1}$  are correctly configured, and we determine which of these are green.

This leaves a set  $W$  of vertices inside  $A_{\alpha-1,\beta+1}$  that have at most four neighbours. If we surround each vertex in  $W$  by a circle of radius 0.5 then we cannot have any point covered by more than 5 of these circles as this means that there is a vertex in  $W$  with at least 5 neighbours. All of these circles are contained in  $C_{\beta+2}$ , which has area  $\pi(\beta+2)^2$ . Therefore

$$|W| \leq \frac{5\pi(\beta+2)^2}{0.5^2\pi} = 20(\beta+2)^2. \quad (3.5)$$

For  $x$  to have any possibility of being 1-pivotal, at this stage there must be a set  $W'$  contained in  $W$  such that if every vertex in  $W'$  is coloured and every vertex in  $W \setminus W'$  is uncoloured then  $x$  becomes 1-pivotal. In this case, with probability at least  $[p(1-p)]^{20(\beta+2)^2}$  we have every vertex in  $W'$  red and every vertex in  $W \setminus W'$  closed, which would imply event  $E'_{n,1}(x)$  occurring. Therefore  $P[E'_{n,1}(x)] \geq [p(1-p)]^{20(\beta+2)^2} P[E_{n,1}(x)]$ .

Now we note that the occurrence or otherwise of  $E'_{n,1}(x)$  is unaffected by the addition or removal of closed vertices in  $A_{\alpha,\beta}(x)$ . This is because the suppression of enhancements in  $A_{\alpha-1,\beta+1}$  means that these added or removed vertices cannot be enhanced themselves, and moreover any vertices they cause to be correctly or incorrectly configured also cannot be enhanced.

Consider creating the marked Poisson process in  $B_n$ , with each Poisson point (vertex)  $x_i$  marked with the pair  $(Y_i, Z_i)$ , in two stages. First, add all marked vertices in  $B_n \setminus A_{\alpha,\beta}(x)$ , and just the red vertices in  $B_n \cap A_{\alpha,\beta}(x)$ . Secondly, add the closed vertices in  $B_n \cap A_{\alpha,\beta}(x)$ . The vertices added at the second stage have no bearing on the event  $E'_{n,1}(x)$ , so  $E'_{n,1}(x)$  is independent of the event that no vertices at all are added in the second stage. Hence,

$$P[E'_{n,1}(x) \cap R_n(x, \alpha, \beta)] \geq \exp(-(1-p)\lambda\pi(\beta^2 - \alpha^2))P[E'_{n,1}(x)],$$

with equality if  $|x| \leq n - \beta$ .

Finally, we use a similar argument to the initial argument in this proof. Suppose  $E'_{n,1}(x) \cap R_n(x, \alpha, \beta)$  occurs. Then there exist at most  $20(\beta + 2)^2$  vertices in  $A_{\beta, \beta+1}(x) \cup A_{\alpha-1, \alpha}(x)$  which are correctly configured for which the possibility of enhancement has been suppressed. If we now allow these to be possibly enhanced, there is a probability of at least  $(1 - q)^{20(\beta+2)^2}$  that none of them is enhanced, in which case the set of coloured vertices is the same for the modified model as for the un-modified model and therefore  $E_{n,1}(x)$  occurs. Taking

$$\delta_1(p, q) = [p(1 - p)(1 - q)]^{20(\beta+2)^2} \exp(-(1 - p)\lambda\pi(\beta^2 - \alpha^2)),$$

we are done.  $\square$

**Proof of Lemma 3.2.** Fix  $p$  and  $q$ . Also fix  $n$  and  $x \in B_n$ , and just write  $P_{n,i}(x)$  for  $P_{n,i}(x, p, q)$ . Define event  $E_{n,1}(x)$  as before, so that  $P_{n,1}(x) = P[E_{n,1}(x)]$ . Also, for  $0 < r < s$  write  $C_r$  for the disc  $C_r(x)$  and  $A_{r,s}$  for the annulus  $A_{r,s}(x)$ . For now we assume  $30.5 < |x| < n - 30.5$ . We create the Poisson process of intensity  $\lambda$  everywhere on  $B_n$  except inside  $C_{30}$ , and decide which of these vertices are red.

Now we create the process of only the red vertices in  $A_{25,30}$  (a Poisson process of intensity  $p\lambda$  in this region). Assuming there will be no closed vertices in  $A_{25,30}$ , we then know which of the closed vertices outside  $C_{30}$  are correctly configured, and we determine which of these are green.

Having done all this, let  $V$  denote the set of current vertices in  $B_n \setminus C_{25}$  that are connected to  $B_{0.5}$  at this stage (by connected we mean connected via a coloured path), and let  $T$  denote the set of current vertices in  $B_n \setminus C_{25}$  that are connected to  $\partial B_n$ .

Let  $N(V)$  be the 1-neighbourhood of  $V$  and let  $N(T)$  be the 1-neighbourhood of  $T$ . Recalling that  $A \triangle B := (A \cup B) \setminus A \cap B$ , we build up the red process inwards (i.e., towards  $x$  from the boundary of  $C_{25}$ ) on  $C_{25} \cap (N(V) \triangle N(T))$  until a red vertex  $y$  occurs (if such a vertex occurs). Set  $r = |y - x|$ . Suppose  $y \in N(V)$  (if instead  $y \in N(T)$  we would reverse the roles of  $V$  and  $T$  in the sequel). Then if  $T \cap C_{r+0.05} \neq \emptyset$  we say that event  $F$  has occurred and we let  $z$  denote an arbitrarily chosen vertex of  $T \cap C_{r+0.05}$ . Otherwise, we build up the red process inwards on  $C_r \cap N(T) \setminus N(V)$  until a red vertex  $z$  occurs (if such a vertex occurs).

Let  $E_2$  be the event that (i) such vertices  $y$  and  $z$  occur, and (ii) the sets  $V$  and  $T$  are disjoint, and (iii)  $|y - z| > 1$ , and (iv) there is no path

from  $y$  to  $z$  through coloured vertices in  $B_n \setminus C_{25}$  that are not in  $V \cup T$ . If  $E_{n,1}(x) \cap R_n(x, 20, 30)$  occurs, then  $E_2$  must occur.

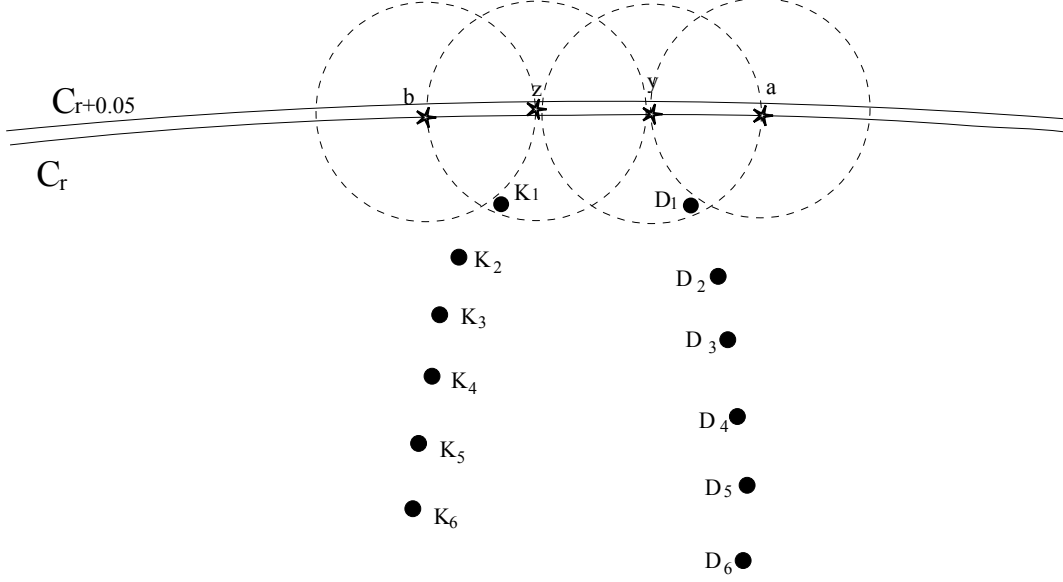


Figure 3: Our convention in the diagrams is to indicate points with lower case letters, and areas with upper case letters. The dashed circles are of radius 1. Here the event  $F$  occurs.

Now suppose  $E_2 \cap F$  has occurred. Let  $a$  be the point (again we use ‘point’ to refer to a point in  $\mathbb{R}^2$ ) which is at distance  $r$  from  $x$  and distance 1 from  $y$  on the opposite side of the line  $xy$  to the side  $z$  is on (see Figure 3). Similarly let  $b$  be the point lying at distance 1 from  $z$  and distance  $r$  from  $x$ , on the opposite side of  $xz$  to  $y$ .

Let  $a_1$  be the point lying inside  $C_r$  at distance 1.01 from  $a$  and 0.99 from  $y$ , and let  $D_1$  be the disc  $C_{0.005}(a_1)$ . Let  $b_1$  be the point at distance 1.01 from  $b$  and 0.99 from  $z$ , and let  $K_1 := C_{0.005}(b_1)$ .

Any red vertex in  $D_1$  will be connected to  $y$  (and therefore to a path to  $B_{0.5}$ ) but cannot be connected to any coloured path to  $\partial B_n$  as  $a$  is the nearest place for such a vertex to be, given  $E_2 \cap F$  occurs. Any red vertex in  $K_1$  will be connected to  $z$  (and therefore a path to  $\partial B_n$ ), but not a path to  $B_{0.5}$ . Also, any vertex in  $D_1$  will be at least 1.1 away from any vertex in  $K_1$ .

Now let  $l$  be the line through  $x$  such that  $a_1$  and  $b_1$  are on different sides of the line and at equal distance from the line. We can pick points  $a_2, a_3, \dots, a_{30}$  such that  $|a_i - a_{i-1}| \leq 0.9$  for  $2 \leq i \leq 30$ , and  $\max(|a_{30} - x|, |a_{29} - x|) \leq 0.9$ ,

but  $|a_i - x| > 1.1$  for  $i \leq 28$ , and none of the  $a_i : i \geq 2$  are within 1 of  $C_r$  or within 0.51 of  $l$  or within 0.01 of another  $a_j$ .

Do the same on the other side of  $l$  with  $b_2, b_3, \dots, b_{30}$ . For  $2 \leq i \leq 30$ , define discs  $D_i := C_{0.005}(a_i)$  and  $K_i := C_{0.005}(b_i)$ .

Let  $I$  be the event that there is at exactly one red vertex in each of the circles  $D_i$  and  $K_i$ ,  $1 \leq i \leq 30$ , and there are no more new vertices anywhere else in  $C_{25}$ , and no closed vertices in  $C_{30} \setminus C_{25}$ . Then

$$P[I|E_2 \cap F] \geq (0.005^2 \pi \lambda p)^{60} \exp(-900\pi\lambda) =: \delta_2.$$

If the events  $E_2$ ,  $F$ ,  $I$  occur and  $Y_0 > p$  then  $x$  is 2-pivotal.

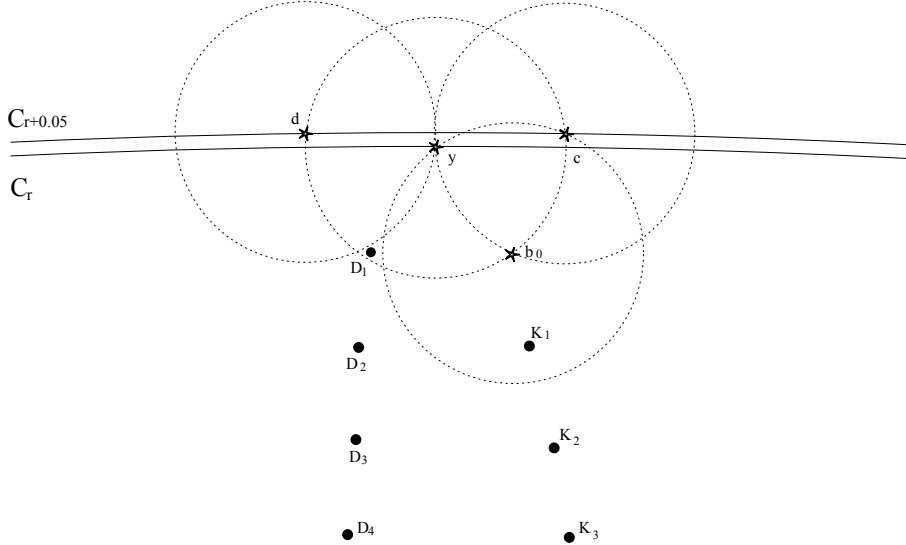


Figure 4: The case where  $F$  does not occur. Here  $b_0$  is the ‘worst possible’ location for  $z$ .

Now we consider the case where  $E_2$  occurs but  $F$  does not, so  $z$  is inside  $C_r$  and is connected to a vertex  $z_1$  in  $T$  that must be outside  $C_{r+0.05}$  because  $T \cap C_{r+0.05} = \emptyset$  (see Figure 4).

Let  $c$  be the point at distance 1 from  $y$  and  $r + 0.05$  from  $x$ , on the same side of the line  $xy$  as  $z$  (assume without loss of generality this is to the right of  $y$ ). This is the closest  $z_1$  can be. Let  $b_0$  be the point inside  $C_r$  at distance 1 from  $y$  and 1 from  $c$ , so this is the furthest left that  $z$  can be. Let  $d$  be the point at distance  $r + 0.05$  from  $x$  and 1 from  $y$ , on the other side of  $y$  to  $c$ . Let  $a_1$  be the point inside  $C_r$  at distance 1.01 from  $d$  and 0.99 from  $y$ , and let

$D_1 := C_{0.005}(a_1)$ . Then any vertex in  $D_1$  is distant at least 1.01 from  $b_0$ , and therefore from  $z$ , as  $z$  cannot be any nearer than  $b_0$ . Also any vertex in  $D_1$  will be at least 1.005 from any other vertices in  $T$ , as  $d$  is the nearest place such a point can be. As before we can then have small discs  $D_2, \dots, D_{30}$  and  $K_1, \dots, K_{30}$  (of radius 0.005) such that having one red vertex in each of these vertices ensures that  $x$  is 2-pivotal.

Given  $E_2 \setminus F$  occurs, the probability of getting 1 red vertex in each of the discs  $D_i$  and  $K_i$  for  $1 \leq i \leq 30$ , and no other new vertices in  $C_{25}$ , and no closed vertices in  $C_{30} \setminus C_{25}$ , is at least  $\delta_2$ . If this happens and also  $Y_0 > p$  then  $x$  is 2-pivotal.

So by Lemma 3.3, the probability that  $x$  is 2-pivotal satisfies

$$\begin{aligned} P_{n,2}(x) &\geq \delta_2(1-p)P[E_2 \cap F] + \delta_2(1-p)P[E_2 \cap F^c] \\ &\geq \delta_2(1-p)P[E_{n,1}(x) \cap R_n(x, 20, 30)] \\ &\geq \delta_1\delta_2(1-p)P_{n,1}(x). \end{aligned}$$

This proves the claim (3.4) for the case with  $30.5 < |x| < n - 30.5$ .

Now suppose  $|x| \leq 30.5$ . Create the Poisson process in  $B_n \setminus C_{40}$ , and decide which of these vertices are red. Then create the red process in  $A_{39,40}(x)$ , and determine which vertices in  $B_n \setminus C_{40}$  are green, assuming there are no closed vertices in  $A_{39,40}(x)$ . Then build up the red process in  $C_{39}$  inwards towards  $x$  until a vertex  $y$  occurs in the process which is connected to  $\partial B_n$ . Let  $H_1$  be the event that such a vertex  $y$  appears at distance  $r$  between 38 and 39 from  $x$ , so  $H_1$  must occur for  $E_{n,1}(x) \cap R_n(x, 20, 40)$  to occur.

If  $x$  is inside  $B_{0.5}$  we can choose points  $a_0$  and  $a_1$  such that they are both outside  $B_{0.5}$ , at distance between 0.8 and 0.9 from  $x$  and at distance between 0.1 and 0.2 from each other. We can then choose  $b_0$  and  $b_1$  such that they are both within 0.9 of  $x$ , further than 1.5 from  $a_0$  and  $a_1$  and between 0.1 and 0.2 from each other. We can then choose points  $a_2, a_3, \dots, a_{100}$  such that  $|a_i - a_{i-1}| \leq 0.9$  for  $2 \leq i \leq 100$ , and  $|a_{100} - y| \leq 0.9$ , no two  $a_i$  are within 0.1 of each other, and no  $a_i$  is within 1.1 of  $x$ ,  $b_0$  or  $b_1$ , or inside  $B_{0.5}$  for  $i \geq 2$ .

Define discs  $D_i = C_{0.05}(a_i)$  and  $K_j = C_{0.05}(b_j)$ . If there is at least one red vertex in each of these discs and no vertices anywhere else in  $C_r$ , and  $Y_0 > p$ , then  $x$  is 2-pivotal. If  $x$  is outside  $B_{0.5}$  we choose points in a similar way but make sure  $b_1$  connects with a path to  $B_{0.5}$ , using little discs  $K_2, K_3, \dots, K_{50}$  which are again of radius 0.05 and are at least 1.1 from the  $a_i$ . Therefore, setting

$$\delta_3 := (1-p)(0.05^2\pi\lambda p)^{152} \exp(-1600\pi\lambda)$$

and using Lemma 3.3, we have for some strictly positive continuous  $\delta_4(p, q)$  that

$$P_{n,2}(x) \geq \delta_3 P[H_1] \geq \delta_3 P[E_{n,1}(x) \cap R_n(x, 20, 40)] \geq \delta_3 \delta_4 P_{n,1}(x).$$

Now suppose  $|x| \geq n - 30.5$ . In this case the proof is similar. Again, create the Poisson process in  $B_n \setminus C_{40}$ . Then create the red process in  $A_{39,40}(x)$  and determine the colours of the vertices in  $B_n \setminus C_{40}$ , assuming there are no closed vertices in  $A_{39,40}(x)$ . Then build the red process in  $C_{39} \cap B_{n-0.5}$  inwards towards  $x$  until a vertex  $y$  occurs that is connected to a path of coloured vertices to  $B_{0.5}$  but not to  $\partial B_n$ . Let  $H_2$  be the event that such a vertex  $y$  occurs at distance  $r$  between 38 and 39 from  $x$ , and that there is no current coloured path from  $B_{0.5}$  to  $\partial B_n$ , so  $H_2$  must occur for  $E_{n,1}(x) \cap R_n(x, 20, 40)$  to occur. Given this vertex  $y$  we can find discs  $D_1, D_2, \dots, D_{100}$  and  $K_1, K_2, \dots, K_{50}$  of radius 0.05 as before such that having a red vertex in each of these discs but no other vertices in  $C_r$  or  $\partial B_n \cap C_{40}$ , and having  $Y_0 > p$ , ensures  $x$  is 2-pivotal. Therefore in this case

$$P_{n,2}(x) \geq \delta_3 P[H_2] \geq \delta_3 P[E_{n,1}(x) \cap R_n(x, 20, 40)] \geq \delta_3 \delta_4 P_{n,1}(x).$$

Take  $\delta(p, q) := \min(\delta_1 \delta_2 (1 - p), \delta_3 \delta_4)$ . By its construction  $\delta$  is strictly positive and continuous in  $p$  and  $q$ , and (3.4) holds for all  $x \in B_n$ , completing the proof of the lemma.  $\square$

The following proposition follows immediately from Lemmas 3.1 and 3.2.

**Proposition 3.2** *There is a continuous function  $\delta : (0, 1)^2 \rightarrow (0, \infty)$  such that for all  $n \geq 100$  and  $(p, q) \in (0, 1)^2$ , we have*

$$\frac{\partial \theta_n(p, q)}{\partial q} \geq \delta(p, q) \frac{\partial \theta_n(p, q)}{\partial p}.$$

**Proof of Theorem 2.1.** Set  $p^* = p_c^{\text{site}}$  and  $q^* = (1/8)(p^*)^2$ . Then using Proposition 3.2 and looking at a small box around  $(p^*, q^*)$ , we can find  $\varepsilon \in (0, \min(p^*/2, 1 - p^*))$  and  $\kappa \in (0, q^*)$  such that for all  $n > 100$  we have

$$\theta_n(p^* + \varepsilon, q^* - \kappa) \leq \theta_n(p^* - \varepsilon, q^* + \kappa).$$

Taking the limit inferior as  $n \rightarrow \infty$ , since  $\theta$  is monotone in  $q$  we get

$$0 < \theta(p^* + \varepsilon, 0) \leq \theta(p^* + \varepsilon, q^* - \kappa) \leq \theta(p^* - \varepsilon, q^* + \kappa).$$

Now set  $p = p^* - \varepsilon$ . Then  $q^* + \kappa \leq p^2$ , so that  $\theta(p, p^2) > 0$ , and by Proposition 3.1, the enhanced model with parameters  $(p, p^2)$  percolates, i.e. has an infinite coloured component, almost surely.

We finish the proof with a coupling argument along the lines of Grimmett and Stacey (1998). Let  $E$  be the set of edges and  $V$  be the set of vertices of  $\mathcal{C}$  (the infinite component). Let  $(X_e : e \in E)$  and  $(Z_v : v \in V)$  be collections of independent Bernoulli random variables with mean  $p$ . From these we construct a new collection  $(Y_v : v \in V)$  which constitutes a (red) site percolation process on  $\mathcal{C}$ , as follows. Let  $e_0, e_1, \dots$  be an enumeration of the edges of  $\mathcal{C}$  and  $v_0, v_1, \dots$  an enumeration of the vertices. Suppose at some point we have defined  $(Y_v : v \in W)$  for some subset  $W$  of  $V$ . Let  $\mathcal{Y}$  be the set of vertices not in  $W$  which are adjacent to some currently active vertex (i.e. a vertex  $u \in W$  with  $Y_u = 1$ ). If  $\mathcal{Y} = \emptyset$  then let  $y$  be the first vertex not in  $W$  and set  $Y_y = Z_y$  and add  $y$  to  $W$ . If  $\mathcal{Y} \neq \emptyset$ , we let  $y$  be the first vertex in  $\mathcal{Y}$  and let  $y'$  be the first currently active vertex adjacent to it, then set  $Y_y = X_{yy'}$  and add  $y$  to  $W$ . Repeating this process builds up the entire red site percolation process, if it does not percolate, or a percolating subset of the red site percolation process if it does percolate. In the latter case, the bond process  $\{X_e\}$  also percolates.

Now suppose the red site process does not percolate. For any correctly configured vertex  $x$  with  $v, w, y, z$  as in Figure 2,  $x$  itself is not red. Therefore at most one edge to  $x$  has been examined, so we can find a first unexamined edge (in the enumeration) to  $v$  or  $w$ , and then to  $y$  or  $z$ . We then declare  $x$  to be green only if both of these edges are open, which happens with probability  $p^2$ . This completes the enhanced site process with  $q = p^2$  and every component in this is contained in a component for the bond process  $\{X_e\}$ .

Therefore, since the enhanced  $(p, p^2)$  site process percolates almost surely, so does the bond process, so  $p_c^{\text{bond}} \leq p < p_c^{\text{site}}$ .  $\square$

## 4 RCM: the key lemma

This section is devoted to stating and proving Lemma 4.1 below, which is the key step in subsequently proving Theorems 2.2 and 2.3. We consider the RCM with connection function  $f : [0, \infty) \rightarrow [0, 1]$ . Throughout this section

we assume that  $f$  is nonincreasing and, moreover, that

$$\sup\{a : f(a) > 0\} = 1. \quad (4.1)$$

Fix  $x \in \mathbb{R}^2$  and (as in the preceding section) for  $r < s$  let  $C_r$  denote the disc of radius  $r$  centred at  $x$  and let  $A_{r,s}$  denote the annulus  $C_s \setminus C_r$ .

We consider the RCM on a Poisson process in  $C_{29}$ , under certain *boundary conditions*, represented by three finite disjoint sets  $V, T$  and  $S$  in  $\mathbb{R}^2 \setminus C_{29}$ , together with a collection  $\mathcal{E}$  of edges amongst the vertices (i.e., elements) of  $S$ . We write  $\mathbf{S}$  for the graph  $(S, \mathcal{E})$  (a subgraph of the complete graph on vertex set  $S$ ). We refer to the triple  $(V, T, \mathbf{S})$  as a *boundary condition*.

In terms of generalising the proof of Theorem 2.1 to the RCM, the set  $V$  (respectively  $T$ ) represents the set of coloured vertices in  $B_n \setminus C_{29}$  that are connected by a coloured path to  $B_{0.5}$  (respectively, to  $\partial B_n$ ), before the vertices inside  $C_{29}$  have been added. The set  $S$  represents the remaining coloured vertices  $B_n \setminus C_{29}$ , and  $\mathcal{E}$  represents the set of edges between these vertices. However, this description is only for motivation, and the present section is self-contained; in particular, no colouring of vertices takes place in this section.

For  $\mu > 0$  and  $0 \leq r < s$ , let  $\mathcal{P}_{\mu,r,s}$  denote a homogeneous Poisson process of intensity  $\mu$  in  $A_{r,s}$ . Given  $(V, T, \mathbf{S})$  as described above, for  $0 \leq r < 29$  the RCM on  $\mathcal{P}_{\mu,r,29}$  with boundary condition  $(V, T, \mathbf{S})$  is obtained as follows: we connect each pair of vertices  $x, y$  with  $x, y \in \mathcal{P}_{\mu,r,29}$  or  $x \in \mathcal{P}_{\mu,r,29}$  and  $y \in V \cup T \cup S$ , by an undirected edge with probability  $f(|x - y|)$ , independently of other pairs. For  $x \in \mathcal{P}_{\mu,r,29}$  we then say  $x$  is *path-connected* to  $T$  (respectively, to  $V$ ) if there is a path from  $x$  to  $T$  (respectively,  $V$ ) using the edges created. If also  $y \in \mathcal{P}_{\mu,r,29}$  then we say  $x$  is path-connected to  $y$  if there is a path from  $x$  to  $y$ , using the edges created along with the edges of  $\mathcal{E}$ .

Let  $V_r$ , respectively  $T_r$  be the set of vertices of  $\mathcal{P}_{\mu,r,29}$  that are path-connected to  $V$ , respectively  $T$ . Let  $S_r$  be the set  $\mathcal{P}_{\mu,r,29} \setminus (V_r \cup T_r)$ . Define the event

$$H(V, T, \mathbf{S}) := \{V_{20} \cap C_{21} \neq \emptyset\} \cap \{T_{20} \cap C_{21} \neq \emptyset\} \cap \{V_{20} \cap T_{20} = \emptyset\}. \quad (4.2)$$

Let  $H'(V, T, \mathbf{S})$  be the intersection of  $H(V, T, \mathbf{S})$  with the event that there exists  $v^* \in C_{20.1}$  and  $t^* \in C_{20.1}$  such that  $|v^* - t^*| > 1.5$  and  $V_{20} \cap C_{20.5} = \{v^*\}$  and  $T_{20} \cap C_{20.5} = \{t^*\}$ , and  $S_{20} \cap C_{20.5} = \emptyset$ . We can now state the main result of this section.

**Lemma 4.1** *Suppose  $f$  is nonincreasing and (4.1) holds. Then there exists a continuous function  $\varepsilon : (0, \infty) \rightarrow (0, \infty)$  such that for any  $\mu \in (0, \infty)$ , and any boundary condition  $(V, T, \mathbf{S})$  we have*

$$P[H'(V, T, \mathbf{S})] \geq \varepsilon(\mu)P[H(V, T, \mathbf{S})]. \quad (4.3)$$

We shall need several further lemmas to prove Lemma 4.1. In these arguments, we often need to build up the Poisson process  $\mathcal{P}_\mu$  in certain regions via a “scanning process”, as described in Meester, Penrose and Sarkar (1997) which gives a rigorous proof that it does indeed build up the Poisson process. For any set of vertices  $U$  and any point  $z \in \mathbb{R}^2$  let  $p(z, U)$  be the probability that a vertex at  $z$  is joined to at least one of the vertices in  $U$ . So  $1 - p(z, U) = \prod_{u \in U} (1 - f(|z - u|))$ .

We shall consider the process  $\mathcal{P}_{\mu, 24, 25}$  as the union of two independent half-intensity processes  $\mathcal{P}_{\mu/2, 24, 25}$  and  $\mathcal{P}'_{\mu/2, 24, 25}$ . Let  $E_1$  be the event that  $\mathcal{P}_{\mu/2, 24, 25}$  has precisely two elements, and one of these is connected to  $V_{25}$  while the other is connected to  $T_{25}$ , and  $V$  is not path-connected to  $T$  through  $\mathcal{P}_{\mu, 25, 29} \cup \mathcal{P}_{\mu/2, 24, 25} \cup S$ .

**Lemma 4.2** *For all boundary conditions  $(V, T, \mathbf{S})$ , it is the case that  $P[E_1] \geq 0.25 \exp(-25\pi\mu)P[H(V, T, \mathbf{S})]$ .*

**Proof.** Create the process  $\mathcal{P}_{\mu, 25, 29}$  and define the sets  $V_{25}$ ,  $T_{25}$  and  $S_{25}$  as described earlier. Then build up an inhomogenous process in from the edge of  $C_{25}$  (i.e. starting at distance 25 from  $x$  and working radially symmetrically inwards) with intensity  $\mu h_1(\cdot)$  where  $h_1(v) = p(v, V_{25})(1 - p(v, T_{25}))$ , until a vertex  $y$  occurs. Then add edges from  $y$  to  $V_{25}$  conditional on there being at least one such edge. Add edges independently from  $y$  to vertices in  $S_{25}$  in the usual way. Do not add any edges from  $y$  to  $T_{25}$ .

Now build up another inhomogenous process in from the edge of  $C_{25}$  with intensity  $\mu h_2(\cdot)$ , where  $h_2(v) = p(v, T_{25})(1 - p(v, V_{25}))$ , until a vertex  $z$  occurs. Add edges from  $z$  conditional on there being at least one edge from  $z$  to  $T_{25}$  and no edge from  $z$  to  $V_{25}$ .

Let  $E'_1$  be the event that we get such vertices  $y$  and  $z$  and  $y$  is not connected to  $z$  through  $S_{25}$ . Then  $E'_1$  must occur for the event  $H(V, T, \mathbf{S})$  to occur.

Let  $E''_1$  be the event that  $E'_1$  occurs with both  $y$  and  $z$  coming from the first half intensity process  $\mathcal{P}_{\mu/2, 24, 25}$  (rather than from  $\mathcal{P}'_{\mu/2, 24, 25}$ ). Then

$P[E_1''|E_1'] = 0.25$ . Given  $E_1''$  occurs, for  $E_1$  to occur we need only that there be no further vertices of  $\mathcal{P}_{\mu/2,24,25}$  besides  $y$  and  $z$ , and the conditional probability of this is at least  $\exp(-49\pi\mu/2)$ . Combining these probability estimates gives the result.  $\square$

Let  $\rho := \inf\{a > 0 : f(a) < 1\}$ , i.e. the radius of certain connection (this could be zero). We shall prove Lemma 4.1 separately for the two cases  $\rho < \frac{1}{\sqrt{2}} - 0.01$  and  $\rho \geq \frac{1}{\sqrt{2}} - 0.01$  (see Lemmas 4.4 and 4.6 below).

Suppose for now that  $\rho < \frac{1}{\sqrt{2}} - 0.01$ . Given  $y, z \in A_{24,25}$  with  $x, y, z$  not collinear, let  $b(y, z)$  be the point at distance 0.999 from  $y$ , at distance  $\rho + 0.01$  from  $xy$  and on the opposite side of the line  $xy$  to  $z$  (see Figure 5). Let  $b(z, y)$  be defined similarly. Define the region

$$Q(y, z) := C_{1.0001}(b(y, z)) \setminus (C_{25} \cup C_\rho(y))$$

and define  $Q(z, y)$  similarly (see Figure 5, where  $Q(z, y)$  is empty). The regions  $Q(y, z)$  and  $Q(z, y)$ , if non-empty, each have diameter less than 0.9 due to  $\rho$  being less than  $\frac{1}{\sqrt{2}} - 0.01$ .

Given  $y$  and  $z$  define  $T_{25}^{y,z}$  and  $V_{25}^{y,z}$  in the same manner as  $T_{25}$  and  $V_{25}$ , respectively, but using the point process  $\mathcal{P}_{\mu,25,29} \cup \{y, z\}$  instead of  $\mathcal{P}_{\mu,25,29}$ .

Suppose  $E_1$  occurs, and let  $y, z$  be the vertices of  $\mathcal{P}_{\mu/2,24,25}$ , with  $y$  path-connected to  $V$  and  $z$  path-connected to  $T$ . Let  $E_2$  be the event that there are no more than two vertices of  $T_{25}^{y,z}$  in  $Q(y, z)$  and no more than two vertices of  $V_{25}^{y,z}$  in  $Q(z, y)$ , and no vertices of  $\mathcal{P}_{\mu,25,29}$  at all, other than those of  $T_{25}^{y,z}$  and  $V_{25}^{y,z}$ .

**Lemma 4.3** *Suppose  $\rho < \frac{1}{\sqrt{2}} - 0.05$ . Then*

$$P[E_2|E_1] \geq f(0.9)^2 \exp(-29^2\pi\mu) =: \varepsilon_1(\mu).$$

**Proof.** The idea here is to condition on what happens inside the annulus  $A_{24,25}$ . The probability  $P[E_1]$  is the product of the probability that there are exactly two vertices in  $\mathcal{P}_{\mu/2,24,25}$ , and the probability that for two uniformly distributed vertices in  $A_{24,25}$ , they are joined one of them to  $T_{25}$  but not  $V_{25}$  and the other to  $V_{25}$  but not  $T_{25}$ . Given  $y$  and  $z$  in  $A_{24,25}$ , let  $I_{y,z}$  be the event (defined in terms of the Poisson process  $\mathcal{P}_{\mu,25,29}$  and associated edges) that  $y \in V_{25}^{y,z} \setminus T_{25}^{y,z}$  and  $z \in T_{25}^{y,z} \setminus V_{25}^{y,z}$ , and let  $p(y, z) = P[I_{y,z}]$  (this also depends on  $V, T$  and  $\mathbf{S}$ ). Then

$$P[E_1] = \exp(-49\pi\mu/2)(\mu/2)^2 \int_{A_{24,25}} \int_{A_{24,25}} p(y, z) dy dz.$$

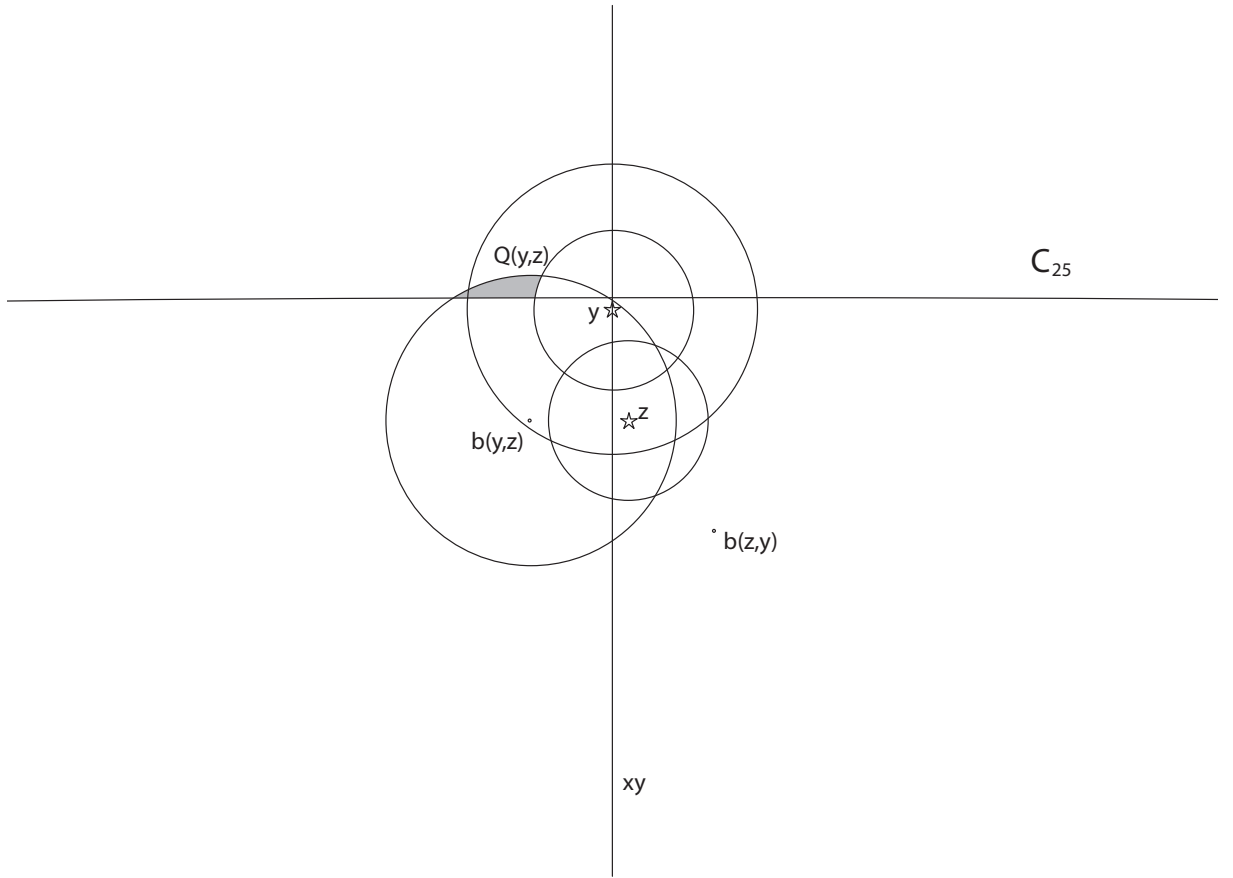


Figure 5: Here is a diagram showing the region  $Q(y, z)$  (in this case  $Q(z, y)$  is empty). The smaller circles are of radius  $\rho$  and the larger ones are of radius 1.0001

Similarly,

$$P[E_1 \cap E_2] = \exp(-49\pi\mu/2)(\mu/2)^2 \int_{A_{24,25}} \int_{A_{24,25}} p'(y, z) dy dz,$$

where  $p'(y, z) = P[I'_{y,z}]$  and  $I'_{y,z}$  is the event that  $I_{y,z}$  occurs and also there are at most two vertices of  $T_{25}^{y,z}$  in  $Q(y, z)$ , and at most two vertices of  $V_{25}^{y,z}$  in  $Q(z, y)$ , and all vertices in  $\mathcal{P}_{\mu,25,29}$  are in  $V_{25,29}^{y,z} \cup T_{25,29}^{y,z}$ . Therefore we just need to show that  $p'(y, z) \geq \varepsilon_1 p(y, z)$  for Lebesgue-almost all  $y, z$  in  $A_{24,25}$ , and for all possible configurations where  $E_1$  occurs. We do this in stages.

**Stage 1.** Fix  $y$  and  $z$ . Let  $V_0 = V \cup \{y\}$  and  $T_0 = T \cup \{z\}$ . We now *exhaustively create* the set of vertices in  $A_{25,29} \setminus Q(z, y)$  that are path-connected to  $V_0$  but not to  $T_0$ , by which we mean the following sequence of steps. First create a process of intensity  $\mu p(\cdot, V_0)(1 - p(\cdot, T_0))$  in  $A_{25,29} \setminus Q(z, y)$ . Add edges from the new vertices to  $V_0 \cup S$  conditional on having at least one edge from each new vertex to  $V_0$  and no edges from the new vertices to  $T_0$ . Let  $V_1$  be the set of vertices outside  $V_0$  that are now path-connected to  $V_0$  (i.e. the newly added vertices and any vertices of  $S$  that are path-connected to them). Next, create a process of intensity  $\mu p(\cdot, V_1)(1 - p(\cdot, V_0))(1 - p(\cdot, T_0))$  in  $A_{25,29} \setminus Q(z, y)$ , and add edges to these points conditional on having at least one edge from each new point to  $V_1$  but no edge to  $V_0$  or  $T_0$ . Let  $V_2$  be the set of points now path-connected to  $V_0$  that were not in  $V_0 \cup V_1$ . Next create a process in  $A_{25,29} \setminus Q(z, y)$  of intensity  $\mu p(\cdot, V_2)(1 - p(\cdot, V_0 \cup V_1))(1 - p(\cdot, T_0))$ .

Continue in this way, at each stage adding those vertices in  $A_{25,29} \setminus Q(z, y)$  that are connected to the latest  $V_i$  but not to earlier sets  $V_{i-1}, \dots, V_0$  or to  $T_0$ . At some stage this procedure must terminate (i.e. the new Poisson process has no points). This completes the exhaustive creation of points that are path-connected to  $V_0$  but not  $T_0$ .

Now let  $V'$  be the union of  $V$  with all vertices path-connected to  $V$  at this stage, and let  $U_y$  be the union of  $\{y\}$  with the set of vertices path-connected to  $y$  at this stage.

**Stage 2.** Next, we exhaustively create (in a similar manner to the above) the set of vertices in  $A_{25,29} \setminus Q(y, z)$  that are path-connected to  $T_0$  but not to  $V' \cup U_y$ . Then let  $T'$  be the union of  $T$  with all vertices path-connected to  $T$  at this stage, and let  $U_z$  be the union of  $\{z\}$  with the set of all vertices path-connected to  $z$  at this stage.

**Stage 3.** Suppose next that  $z \notin T'$ . Otherwise, go on to Stage 4 below. Then, since we have exhaustively created the vertices connected to  $T' \cup U_z$

outside  $Q(y, z)$ , for  $I_{y,z}$  to occur there must be a vertex in  $Q(y, z)$  connected to  $T'$  and a vertex (possibly the same one) in  $Q(y, z)$  connected to  $U_z$ . Build up the process in  $Q(y, z)$  towards  $x$  with intensity

$$\mu p(\cdot, U_z) p(\cdot, T') [1 - p(\cdot, V' \cup U_y)]$$

until we get a vertex  $u$  (if any). If such a vertex occurs then we add edges from  $u$  to  $U_z$  and to  $T'$  conditional on there being at least one of each type, and add no edges from  $u$  to  $V' \cup U_y$ . We then let  $T'' := T' \cup U_z \cup \{u\}$ , and go to Stage 4 below.

If  $u$  does not occur, build up two more processes in  $Q(y, z)$ , with intensities

$$\mu [1 - p(\cdot, U_z)] p(\cdot, T') [1 - p(\cdot, V' \cup U_y)]$$

and

$$\mu p(\cdot, U_z) [1 - p(\cdot, T')] [1 - p(\cdot, V' \cup U_y)]$$

until we get vertices  $u_1$  and  $u_2$  respectively. If we get such vertices then  $u_1$  will be joined to  $T'$  and  $u_2$  will be joined to  $U_z$ . Also,  $u_1$  will be joined to  $u_2$  with probability at least  $f(0.9)$ . Assume this happens (so now  $z$  is path-connected to  $T$ ), and let  $T'' := T' \cup U_z \cup \{u_1, u_2\}$  and go to Stage 4. If we do not get  $u_1$  and  $u_2$ , then  $I_{y,z}$  cannot occur.

**Stage 4.** Suppose now that  $y \notin V'$ . Otherwise, go on to Stage 5 below. Build up the process in  $Q(z, y)$  towards  $x$  with intensity

$$\mu p(\cdot, U_y) p(\cdot, V') [1 - p(\cdot, T'' \cup U_z)]$$

until we get a vertex  $w$ . If such a vertex occurs, then add edges from  $w$  to  $U_y$  and to  $V'$  conditional on there being at least one of each type, add none to  $T'' \cup U_z$ . We now have a path from  $y$  to  $V$  and go to Stage 5 below.

If  $w$  does not occur, build up two more processes in  $Q(z, y)$ , with intensities

$$\mu [1 - p(v, U_y)] p(v, V') [1 - p(v, T'' \cup U_z)]$$

and

$$\mu p(v, U_y) [1 - p(v, V')] [1 - p(v, T'' \cup U_z)]$$

until we get vertices  $w_1$  and  $w_2$  respectively. If we get such vertices, then  $w_1$  will be joined to  $V'$  and  $w_2$  will be joined to  $U_y$ . Also  $w_1$  will be joined to  $w_2$  with probability at least  $f(0.9)$ . Assume this happens (so then we have a

path from  $y$  to  $V$ ), and go to Stage 5. If  $w_1$  and  $w_2$  do not occur, then  $I_{y,z}$  cannot occur.

**Stage 5.** By now we have  $y$  connected (by a path) to  $V$  and  $z$  connected to  $T$ , and  $V$  not connected to  $T$ . Now sample the rest of  $\mathcal{P}_{\mu,25,29}$ . Then as long as no more vertices occur when we do this (an event with probability at least  $\exp(-29^2\pi\mu)$ ), event  $I_{y,z}$  occurs. Therefore, we have shown that  $p'(y, z) \geq \varepsilon_1 p(y, z)$ , as required.  $\square$

**Lemma 4.4** *Suppose that  $f$  is nonincreasing and (4.1) holds, and that  $\rho < \frac{1}{\sqrt{2}} - 0.05$ . Then the conclusion of Lemma 4.1 holds.*

**Proof.** Suppose  $E_1 \cap E_2$  occurs, and let  $y$  and  $z$  be as in the definition of  $E_1$  (i.e. the points in  $\mathcal{P}_{\mu/2,24,25}$  that are path-connected to  $V$  and to  $T$  respectively). Let  $b_1 = b(y, z)$  and  $a_1 = b_1(z, y)$ . Define discs  $D_1 := C_{0.0001}(b_1)$  and  $K_1 := C_{0.0001}(a_1)$ . Then

$$\min(\text{dist}(D_1, z), \text{dist}(K_1, y), \text{dist}(D_1, K_1)) \geq \rho + 0.005; \quad (4.4)$$

$$\min(\text{dist}(D_1, \mathbb{R}^2 \setminus C_{25}), \text{dist}(K_1, \mathbb{R}^2 \setminus C_{25})) \geq \max(\rho + 0.005, 0.6), \quad (4.5)$$

and for any  $b' \in D_1$  and  $a' \in K_1$  we have  $\max(|b' - y|, |a' - z|) \leq 0.9991$ .

Next take further discs  $D_i = C_{0.0001}(b_i)$  and  $K_i = C_{0.0001}(a_i)$ , for  $2 \leq i \leq 7$ , such that each of these discs is contained in  $A_{20,24}$ , and discs  $D_1, K_1, \dots, D_7, K_7$  are disjoint, and

$$\begin{aligned} |b_i - b_{i-1}| &= |a_i - a_{i-1}| = 0.999, \quad 2 \leq i \leq 7; \\ \min(\text{dist}(D_2, K_1), \text{dist}(K_2, D_1)) &\geq \rho + 0.005; \\ \text{dist}(D_i, K_j) &\geq 1.1, \quad 1 \leq i, j \leq 7, (i, j) \notin \{(1, 1), (1, 2), (2, 1)\}; \\ \min(|b_i - x|, |a_i - x|) &\geq 20.6, \quad 2 \leq i \leq 6; \end{aligned}$$

and  $|b_7 - x| = |a_7 - x| = 20.05$  and  $|b_7 - a_7| \geq 1.5$ .

Now create the Poisson process  $\mathcal{P}'_{\mu/2,24,25} \cup \mathcal{P}_{\mu,20,24}$ . Let  $E_3$  be the event that we get exactly one new vertex in each of  $D_i$  and  $K_i$  (denoted  $y_i$  and  $z_i$  respectively) for  $1 \leq i \leq 7$ , and no other new vertices. Then

$$P[E_3 | E_1 \cap E_2] \geq ((0.0001)^2 \pi \mu / 2)^{14} \exp(-25^2 \pi \mu) =: \varepsilon_2. \quad (4.6)$$

Now, assuming  $E_1 \cap E_2 \cap E_3$  occurs, decide which edges occur involving the new vertices. The probability that we get edges forming the paths  $(y, y_1, y_2, \dots, y_7)$  and  $(z, z_1, \dots, z_7)$  is at least  $f(0.9991)^{14}$ .

By (4.4), the probability that  $y_1$  is not joined to  $z$ ,  $z_1$  or  $z_2$  is at least  $[1 - f(\rho + 0.005)]^3$ . Also by (4.5), the probability that  $y_1$  is joined to no vertices of  $T_{25}^{y,z} \cap A_{25,29}$  is at least  $[1 - f(\rho + 0.005)]^2$ , because at most 2 such vertices lie in  $Q(y, z)$  since event  $E_2$  is assumed to occur, and no such vertices lie within  $\rho$  of  $y$  since event  $E_1$  is assumed to occur, and all other such vertices are more than unit distance from  $y_1$ .

Similarly,  $z_1$  is not connected to  $y$  or  $y_2$  or any vertex in  $T_{25}^{y,z} \cap A_{25,29}$  with probability at least  $[1 - f(\rho + 0.005)]^4$  given  $E_1 \cap E_2$ .

If  $y_2$  is not connected to  $z_1$  and  $z_2$  is not connected to  $y_1$ , then for  $2 \leq i \leq 7$ , none of the vertices  $y_i$  can be connected to any of the vertices  $z_j$  or to  $T_{25}^{y,z} \cap A_{25,29}$ , and none of the vertices  $z_i$  can be connected to any of the vertices  $y_j$  or to  $V_{25}^{y,z} \cap A_{25,29}$ . Therefore, we arrive at

$$P[H'(T, V, \mathbf{S}) | E_1 \cap E_2 \cap E_3] \geq f(0.9991)^{14} [1 - f(\rho + 0.005)]^9 := \varepsilon_3.$$

Hence, by (4.6) and Lemmas 4.2 and 4.3, taking  $\varepsilon = 0.25 \exp(-25\pi\mu) \varepsilon_1 \varepsilon_2 \varepsilon_3$ , we have the desired result (4.3) for  $\rho \leq \frac{1}{\sqrt{2}} - 0.05$ .  $\square$

Now, to complete the proof of Lemma 4.1 we look at the case where  $\rho \geq \frac{1}{\sqrt{2}} - 0.05$ . We create the process  $\mathcal{P}_{\mu,25,29}$  and define  $V_{25}, T_{25}$  and  $S_{25}$  as before. Let  $E_4$  be the event that  $V_{25}$  and  $T_{25}$  are disjoint. This must occur for  $E_1$  to occur.

We then add the half intensity process  $\mathcal{P}_{\mu/2,24,25}$ . Let  $F_V$  be the event that  $E_4$  occurs and there is just one vertex  $y$  of  $\mathcal{P}_{\mu/2,24,25}$ , and it is connected to  $V_{25}$  but not  $T_{25}$ , and  $T_{25}$  includes a vertex in  $A_{|y-x|,|y-x|+.05}$ . Similarly, let  $F_T$  be the event that  $E_4$  occurs and there is just one vertex  $y$  of  $\mathcal{P}_{\mu/2,24,25}$ , and it is connected to  $T_{25}$  but not  $V_{25}$ , and  $V_{25}$  includes a vertex in  $A_{|y-x|,|y-x|+.05}$ .

Let  $G_V$  be the event that  $E_4$  occurs and there are just two vertices  $y, z$  of  $\mathcal{P}_{\mu/2,24,25}$ , and  $y$  is connected to  $V_{25}$  but not  $T_{25}$  and  $z$  is connected to  $T_{25}$  but not  $V_{25}$ , and  $|y-x| > |z-x|$  and  $T_{25} \cap A_{|y-x|,|y-x|+.05} = \emptyset$ . Similarly let  $G_T$  be the event that  $E_4$  occurs and there are two vertices  $y, z$  of  $\mathcal{P}_{\mu/2,24,25}$ , and  $y$  is connected to  $T_{25}$  but not  $V_{25}$  and  $z$  is connected to  $V_{25}$  but not  $T_{25}$ , and  $|y-x| > |z-x|$  and  $V_{25} \cap A_{|y-x|,|y-x|+.05} = \emptyset$ .

**Lemma 4.5** *Let  $\varepsilon_4(\mu) := 0.25 \exp(-25\pi\mu)$ . Then for any boundary conditions  $(V, T, \mathbf{S})$  we have*

$$P[H(V, T, \mathbf{S})] \leq \varepsilon_4^{-1} (P[F_V] + P[F_T] + P[G_V] + P[G_T]). \quad (4.7)$$

**Proof.** After creating the process  $\mathcal{P}_{\mu,25,29}$ , we build a process of intensity

$$\mu(p(\cdot, V_{25})(1 - p(\cdot, T_{25})) + p(\cdot, T_{25})(1 - p(\cdot, V_{25})))$$

inwards into  $C_{25}$ , until we get a vertex  $y \in A_{24,25}$ . Let  $E'$  be the event that such a vertex occurs. Event  $E'$  must occur for  $H(V, T, \mathbf{S})$  to occur.

If  $E'$  occurs, add edges from  $y$  to  $V_{25} \cup T_{25} \cup S_{25}$ , conditional on there being at least one edge from  $y$  to  $V_{25} \cup T_{25}$  but there not being edges from  $y$  both to  $T_{25}$  and to  $V_{25}$ .

Suppose for now that  $y$  is connected to  $V_{25}$  (we call this event  $E'_V$ ). Let  $F'_V$  be the event that there is a vertex of  $T_{25}$  in the thin annulus  $A_{|y-x|, |y-x|+0.05}$ . If  $F'_V$  occurs, then if  $y$  comes from the first half-intensity process  $\mathcal{P}_{\mu/2,24,25}$  and there are no further vertices from  $\mathcal{P}_{\mu/2,24,25}$  (an event of probability at least  $\varepsilon_4$ ), event  $F_V$  occurs.

If  $F'_V$  does not occur, then let  $V_{25}^y$  denote the set of points of  $\mathcal{P}_{\mu,25,29} \cup \{y\}$  that are path-connected to  $V$ , and build a process of intensity  $\mu p(\cdot, T_{25})(1 - p(\cdot, V_{25}^y))$ , inwards inside  $C_{|y-x|}$ , until we get a vertex  $z \in A_{24,|y-x|}$  (this must happen if  $E'_V \cap H(V, T, \mathbf{S})$  is to occur but  $F'_V$  does not occur). If then  $y$  and  $z$  both come from  $\mathcal{P}_{\mu/2,24,25}$  and there are no further vertices in  $\mathcal{P}_{\mu/2,24,25}$  (an event of probability at least  $\varepsilon_4$ ), then  $G_V$  occurs. Combining these yields

$$\begin{aligned} P[F_V] + P[G_V] &\geq \varepsilon_4(P[E'_V \cap F'_V] + P[E'_V \cap H(V, T, \mathbf{S}) \setminus F'_V]) \\ &\geq \varepsilon_4 P[E'_V \cap H(V, T, \mathbf{S})]. \end{aligned}$$

If  $E' \setminus E'_V$  occurs, then  $y$  is connected to  $T_{25}$  and a similar argument yields

$$P[F_T] + P[G_T] \geq \varepsilon_4 P[(E' \setminus E'_V) \cap H(V, T, \mathbf{S})],$$

and combining the last two estimates gives us (4.7).  $\square$

The following result, combined with Lemma 4.4, completes the proof of Lemma 4.1.

**Lemma 4.6** *Suppose that  $f$  is nonincreasing and (4.1) holds, and that  $\rho \geq \frac{1}{\sqrt{2}} - 0.05$ . Then the conclusion of Lemma 4.1 holds.*

**Proof.** If  $F_V$  or  $F_T$  occurs we can continue in similar fashion to the argument for Gilbert's graph, as follows. Suppose  $F_V$  occurs, let  $y$  be as in the definition of  $F_V$  and set  $r = |y - x|$ , and let  $z$  be an arbitrarily chosen point of  $T_{25}$  lying in  $A_{r,r+0.05}$ .

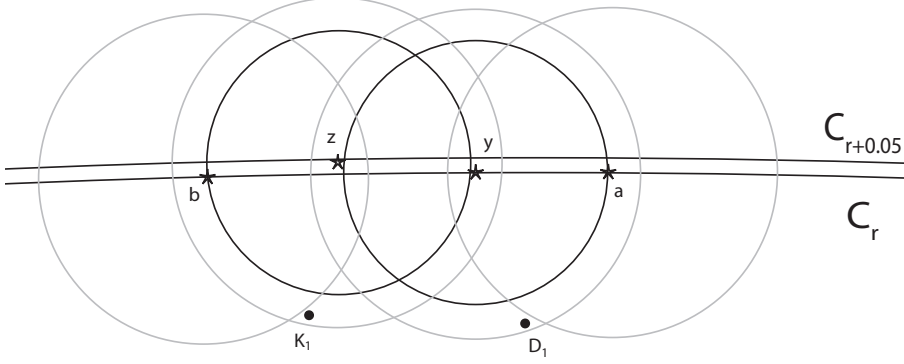


Figure 6: The grey circles are of radius 1 and the black circles are of radius  $\rho$ .

Let  $a$  be the point with  $|a - y| = \rho$  and  $|a - x| = r$ , on the other side of  $y$  to  $z$  (see Figure 6). Let  $b$  be the point with  $|b - z| = \rho$  and  $|b - x| = r$ , lying on the other side of  $z$  to  $y$ . Let  $a_1$  be the point in  $C_r$  with  $|a_1 - y| = 0.99$  and  $|a_1 - a| = 1.01$  and let  $D_1 := C_{0.001}(a_1)$ . Similarly let  $b_1$  be the point in  $C_r$  with  $|b_1 - z| = 0.99$  and  $|b_1 - b| = 1.01$ , and let  $K_1 := C_{0.001}(b_1)$ . Note that  $|y - z| > \rho$  so  $\text{dist}(D_1, K_1) > \rho + 0.01$ .

Let  $D_2, \dots, D_7$  and  $K_2, \dots, K_7$  be discs of radius 0.001 and successive centres distant 0.99 from each other, such that, as before, having exactly one red vertex in each of these little circles and no other vertices in  $A_{20,25}$ , and connections between the vertices in successive circles  $D_i, D_{i+1}$  and  $K_j, K_{j+1}$  ensures that  $H'(V, T, \mathbf{S})$  occurs.

Now sample  $\mathcal{P}'_{\mu/2,24,25} \cup \mathcal{P}_{\mu,20,24}$  and consider the event  $E_5$ , that there is exactly one new vertex  $y_i$  in  $D_i$  and exactly one new vertex  $z_i$  in  $K_i$  for  $1 \leq i \leq 7$ , and no other new vertices. Then

$$P[E_5 | F_V] \geq (0.001^2 \pi \mu)^{14} \exp(-25^2 \pi \mu) =: \varepsilon_5.$$

Next, decide which edges are created from the new vertices. We want  $y_1$  to connect with  $y$  (which happens with probability at least  $f(0.991)$ ) but not to any vertices in  $T_{25}^{y,z}$  (which cannot happen as  $a$  is the closest place for a vertex in  $T_{25}^{y,z}$ ). Similarly we also want  $z_1$  to connect with  $z$  but not to  $V_{25}^{y,z}$ .

Also we want  $z_1, y_1$  to not to be joined, and we want connections between vertices in successive circles  $D_i, D_{i+1}$  and  $K_i, K_{i+1}$ . Given  $F_V \cap E_5$ , these events all happen with probability at least  $f(0.991)^{14}(1 - f(\rho + 0.01))$ , in which case  $H'(V, T, S)$  occurs; hence

$$P[H'(V, T, \mathbf{S})|F_V] \geq f(0.991)^{14}[1 - f(\rho + 0.01)]\varepsilon_5 := \varepsilon_6 \quad (4.8)$$

and similarly  $P[H'(V, T, \mathbf{S})|F_T] \geq \varepsilon_6$ .

Now suppose the event  $G_V$  occurs. Then with  $r := |y - x|$ , we have  $z$  inside  $C_r$  connected to a vertex  $z_0$  of  $T_{25}$  which must be outside  $C_{r+0.05}$ .

Let  $a$  be the point with  $|a - y| = \rho$  and  $|a - x| = r + 0.05$  on the opposite side of  $y$  to  $z_0$ . Let  $l_a$  be the arc of  $C_{r+0.05}$  to the left of  $a$  (see Figure 7). Let

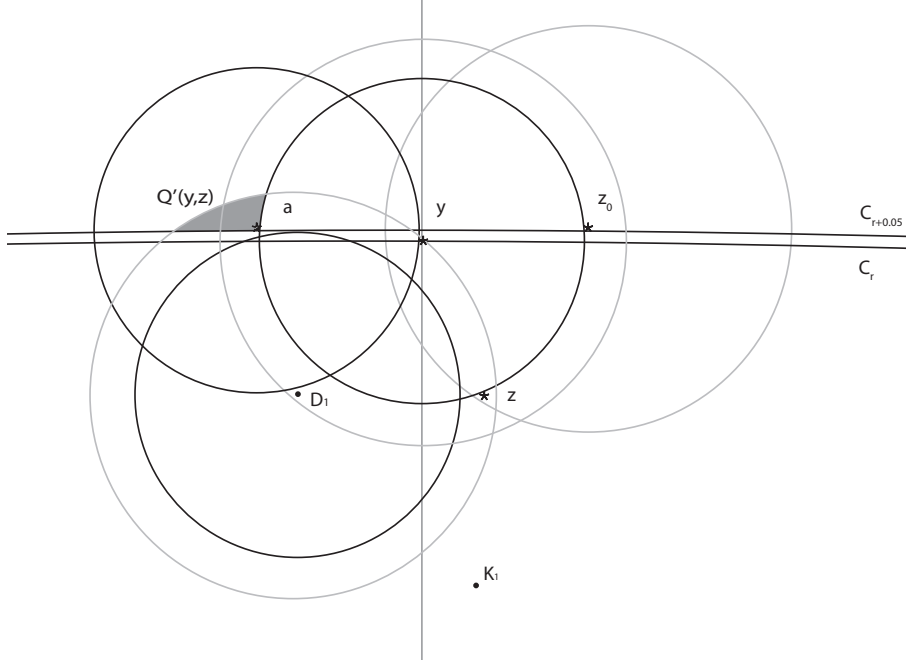


Figure 7: The grey circles are of radius 1.0001 and the black circles are of radius  $\rho$ .

$b'(y, z)$  be the point at distance 0.999 from  $y$  and  $\rho + 0.01$  from  $l_a$ , and define the region

$$Q'(y, z) := C_{1.0001}(b'(y, z)) \setminus (C_\rho(y) \cup C_{r+0.05}(x)).$$

The diameter of  $Q'(y, z)$  is less than 0.9, due to  $\rho$  being at least  $2^{-1/2} - 0.05$ . Let  $E_6$  be the event that there are no more than 2 vertices of  $T_{25}^{y,z}$  in  $Q'(y, z)$ , and no other vertices than those of  $T_{24}^{y,z} \cup V_{24}^{y,z}$  in  $A_{25,29}$ . By a similar argument to the proof of Lemma 4.3 the conditional probability of  $E_6$  satisfies

$$P[E_6|G_V] \geq f(0.9) \exp(-29^2\mu) =: \varepsilon_7. \quad (4.9)$$

Set  $D_1 = C_{0.0001}(b'(y, z))$ . If there is a vertex in  $D_1$  it will be distant at least  $\rho + 0.001$  from  $z$  and from any vertex in  $T_{25}^{y,z}$  (as  $l_a$  is the closest place such a vertex can be given  $G_V$  occurs) and at most 0.9991 from  $y$ . Let  $a_1$  be the point distant 0.999 from  $z$  on the line parallel with  $xy$  through  $z$ , and let  $K_1 := C_{0.0001}(a_1)$ . We can then pick little discs  $D_i, K_i$ ,  $2 \leq i \leq 7$ , of radius 0.0001, as before (see Figure 4) such that if there is exactly one vertex in each of these discs and no other vertices in the rest of  $\mathcal{P}_{20,25}$ , and connections between vertices in successive discs, then  $H'(V, T, S)$  occurs.

Suppose that for the Poisson process  $\mathcal{P}_{\mu,20,24} \cup \mathcal{P}'_{\mu/2,24,25}$  there is exactly one vertex  $y_i \in D_i$  and exactly one vertex  $z_i \in K_i$  for each  $i$  and there are no other vertices. This has probability at least  $[0.0001^2\pi\mu/2]^{14} \exp(-25^2\pi\mu)$ . Given this event, consider now the event that we get all connections occurring along the paths  $(y, y_1, \dots, y_7)$  and  $(z, z_1, \dots, z_7)$  but no connection from  $y_1$  to any vertex in  $T_{25}^{y,z}$ . This has probability at least  $f(0.9991)^{14}[1 - f(\rho + 0.001)]^3$  (assuming  $E_6$  occurs), and if this occurs then  $H'(V, T, \mathbf{S})$  occurs. Hence,  $P[H'(V, T, \mathbf{S})|E_6 \cap G_V] \geq \varepsilon_8$ , with

$$\varepsilon_8 := [0.0001^2\pi\mu/2]^{14} \exp(-25^2\pi\mu) f(0.9991)^{14} [1 - f(\rho + 0.001)]^3 \leq \varepsilon_6.$$

Combined with (4.8) and (4.9), and a similar argument in the case of  $G_T$ , this gives us (for  $\rho \geq \frac{1}{\sqrt{2}} - 0.05$ ) the bound

$$4P[H'(V, T, \mathbf{S})] \geq \varepsilon_7\varepsilon_8(P[G_V] + P[G_T] + P[F_V] + P[F_T]).$$

Combined with Lemma 4.5, this gives us the desired result (4.3) with  $\varepsilon = 0.25\varepsilon_4\varepsilon_7\varepsilon_8$ .  $\square$

## 5 Proof of Theorem 2.2

We now generalise Theorem 2.1 to the random connection model with connection function  $f : [0, \infty) \rightarrow [0, 1]$ , where  $f$  is nonincreasing and has bounded

support. Without loss of generality we assume (4.1) holds (as if not we can rescale). For the enhancement this time we say that a vertex  $x$  is correctly configured if it is closed and there are only 4 vertices  $v, w, y, z$  within 1 of  $x$ , they are all red and joined to  $x$  and  $v \sim w$  and  $y \sim z$  but there are no other edges amongst  $v, w, y, z$ . Notice that another vertex could be not joined to  $x$  but still cause it to be incorrectly configured by being within 1 of it.

All parts of the proof for this model are the same apart from Lemma 3.2. Accordingly we give a proof of the equivalent of Lemma 3.2 for the Random Connection Model under our current assumptions.

**Lemma 5.1** *Suppose  $f$  is nonincreasing and (4.1) holds. Then there is a continuous function  $\delta : (0, 1)^2 \rightarrow (0, \infty)$  such that for all  $(p, q) \in (0, 1)^2$ ,  $n > 100$  and  $x \in B_n$ ,*

$$P_{n,2}(x) > \delta(p, q)P_{n,1}(x).$$

In the proof we again write  $C_r$  for  $C_r(x)$ . Also we define events  $E_{n,1}(x)$  and  $R_n(x, \alpha, \beta)$  as in Section 3. It can easily be seen that the proof of Lemma 3.3 extends to this case as again the number of possible green vertices in the completed process in a bounded region is bounded. Therefore

$$P[E_{n,1}(x) \cap R_n(x, 20, 30)] \geq \delta_1 P_{n,1}(x). \quad (5.1)$$

Assume for now that  $30.5 < |x| < n - 30.5$ . Now suppose we create the whole process of intensity  $\lambda$  in  $B_n \setminus C_{30}$  and the red process of intensity  $p\lambda$  in the annulus  $A_{29,30}$ . We decide which vertices outside  $C_{30}$  are red, and assuming no closed vertices occur in  $A_{29,30}$ , we then know which vertices outside  $C_{30}$  are correctly configured. We then determine which of these are green.

At this stage, let  $V$  be the set of coloured vertices in  $B_n \setminus C_{29}$  that are connected (by a coloured path) to  $B_{0.5}$  and let  $T$  be the coloured vertices in  $B_n \setminus C_{29}$  that are connected to  $\partial B_n$ . Let  $S$  be the remaining coloured vertices in  $B_n \setminus C_{29}$ , and let  $\mathcal{E}$  be the set of edges on  $S$  inherited from the original random connection model. Set  $\mathbf{S} := (S, \mathcal{E})$ .

Then we can apply Lemma 4.1, using these boundary conditions, to the Poisson process of red vertices, of intensity  $\mu = \lambda p$  inside  $C_{29}$ . If  $E_{n,1}(x) \cap R_n(x, 20, 30)$  occurs, then  $H(V, T, \mathbf{S})$  must occur, and therefore by Lemma 4.1,

$$P[H'(V, T, \mathbf{S})] \geq \varepsilon(\lambda p)\delta_1 P_{n,1}(x).$$

Now we can find  $\delta_2$  such that given  $H'(V, T, \mathbf{S})$  occurs, the probability of  $x$  being 2-pivotal is at least  $\delta_2$ . Indeed, with  $y^*$  and  $z^*$  as in the definition

of  $H'(V, T, S)$ , we just find little discs  $D_1, \dots, D_{30}$  and  $K_1, \dots, K_{30}$  of radius 0.005 leading from  $y^*$  and  $z^*$  in towards a bow-tie configuration around  $x$  such that having one red vertex in each of these discs, with connections between successive discs, and no other vertices inside  $C_{20}$ , no vertices in the non-red process inside  $C_{30}$ , and having  $Y_0 > p$  ensures  $x$  is 2-pivotal. This all occurs with probability at least

$$\delta_2 := (0.005^2 \pi \lambda p)^{60} [f(0.9)]^{64} \exp(-900\pi\lambda)(1-p).$$

Therefore we have

$$P_{n,2}(x) \geq \delta_1 \delta_2 \varepsilon P_{n,1}(x)$$

for  $30.5 < |x| < n - 30.5$ .

If  $|x| \leq 30.5$  or  $|x| \geq n - 30.5$ , then by minor modifications of the last part of the proof of Lemma 3.2 we can find some continuous  $\delta_3 : (0, 1)^2 \rightarrow (0, \infty)$  such that  $P_{n,2}(x) \geq \delta_3(p, q) P_{n,1}(x)$ . So taking  $\delta = \delta_1 \delta_2 \delta_3 \varepsilon$  will give us the result.  $\square$

## 6 Proof of Theorems 2.3 and 2.4

For proving Theorem 2.3, it is useful to consider mixed site-bond percolation on the graph  $RCM(\lambda, f)$ . Each site is open with probability  $p$ , and each bond is open with probability  $q$ . Clearly the graph resulting from performing this mixed percolation process on  $RCM(\lambda, f)$  may be viewed as a realisation of  $RCM(p\lambda, qf)$ .

In proving Theorem 2.3 we assume without loss of generality that (4.1) holds. We consider a new site percolation model, where sites are open with probability  $pq$  if they are correctly configured and with probability  $p$  if they are not correctly configured. Each site is designated either an *up-site* or a *down-site*, each with probability  $1/2$ , independently of everything else. We say vertex  $y$  is a 1-neighbour of vertex  $x$  if  $|y - x| \leq 1$ . A vertex  $x$  is correctly configured if it has exactly two 1-neighbours (denoted  $y_1$  and  $y_2$ , say) and  $x$  is connected both to  $y_1$  and to  $y_2$ , and  $x$  is a down-site but  $y_1$  and  $y_2$  are up-sites (see Figure 8).

The extra randomization of up-sites and down-sites is designed to ensure that if a site is correctly configured, then its neighbours are not.

We build this model by having a Poisson process of intensity  $\lambda$  and labelling vertices  $x_1, x_2, \dots$  in order of distance from the origin. We also have

independent uniform random variables  $W_i, Y_i, Z_i$  for  $i = 0, 1, 2, \dots$ . We say vertex  $x_i$  is up-site if and only if  $W_i < 1/2$ . If a vertex  $x_i$  is correctly configured it is open if  $Y_i < p$  and  $Z_i < q$ . Otherwise it is open if  $Y_i < p$ . We define  $\partial B_n$  to be  $B_n \setminus B_{n-0.2}$ . We let  $A_n$  be the event that there is an open path from  $B_{0.2}$  to  $\partial B_n$  in the process restricted to  $B_n$ , and for  $x \in B_n$  define  $A_n^x$  similarly in terms of the process in  $B_n$  with an added vertex at  $x$ .

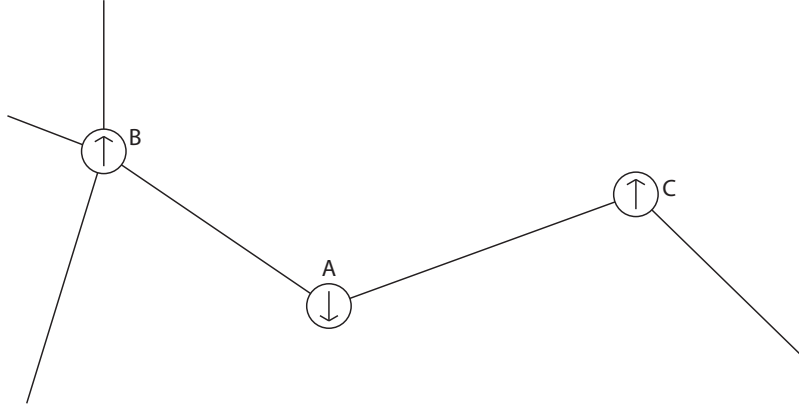


Figure 8: Here the vertex  $A$  is correctly configured but  $B$  and  $C$  are not.

Let the sites  $x_i$  for which  $Y_i < p$  be denoted *red* (a standard Bernoulli site percolation process). The set of open sites may be viewed as a *diminishment* of the set of red sites, in which each correctly configured red site is removed with probability  $1 - q$ . We can couple the diminished site percolation process to the mixed site-bond process (with parameters  $p, q$ ) in such a way that if the mixed process percolates then so does the diminished site process, as follows.

List the edges of this graph in an arbitrary order as  $e_1, e_2, \dots$ , and determine the open sites and edges for the mixed site-bond process. Deem each vertex to be red if and only if it is open in the mixed process. If a vertex  $x$  is correctly configured, then it has degree 2 and has no correctly configured neighbour. In this case, let  $x$  be diminished (i.e. removed from the set of red vertices) if and only if the first edge incident to it (according to the given ordering) is closed.

If there is an infinite open path in the mixed percolation process, we can find such a path which starts at a non-correctly configured vertex. In this

case, every vertex in the path will be red and undiminished, so there will be an infinite path in the diminished site percolation process as well.

Let  $\theta_n(p, q)$  be the probability that  $A_n$  occurs and let  $\theta(p, q)$  be the limit inferior. The proof of Proposition 3.1 is easily modified to this model. We say that a point  $x$  is 3-pivotal if putting a vertex at  $x$  and making  $Y_0 < p$  means that  $A_n^x$  occurs but having  $Y_0 > p$  means it does not. Similarly with 4-pivotal and  $Z_0$ . Again we have a form of the Margulis-Russo formulae:

$$\frac{\partial \theta_n(p, q)}{\partial p} = \int_{B_n} \lambda P_{n,3}(x, p, q) dx \quad (6.1)$$

and

$$\frac{\partial \theta_n(p, q)}{\partial q} = \int_{B_n} \lambda P_{n,4}(x, p, q) dx. \quad (6.2)$$

We then need to prove the equivalent of Lemma 3.2:

**Lemma 6.1** *There is a continuous function  $\delta : (0, 1)^2 \rightarrow (0, \infty)$  such that for all  $n > 100$  and all  $x \in B_n$  we have*

$$P_{n,4}(x, p, q) \geq \delta(p, q) P_{n,3}(x, p, q). \quad (6.3)$$

Before proving this, we give the equivalent of Lemma 3.3 which says we can assume all the vertices in an annulus of fixed size are red and none of them are diminished. Given  $p$  and  $q$ , and given  $2 < \alpha < \beta$ , let  $R_n(x, \alpha, \beta)$  be the event that all vertices in  $A_{\alpha, \beta}(x)$  are red. Let  $R'_n(x, \alpha, \beta)$  be the event that  $R_n(x, \alpha, \beta)$  occurs and also none of the vertices in  $A_{\alpha-1, \beta+1}(x)$  is diminished. We claim there exists continuous  $\delta_1 : (0, 1)^2 \rightarrow (0, \infty)$  such that for all  $n$  and  $x$  we have

$$P[E_{n,3}(x) \cap R'_n(x, \alpha, \beta)] \geq \delta_1(p, q) P_{n,3}(x, p, q). \quad (6.4)$$

To prove this we create the whole Poisson point process of intensity  $\lambda$  in  $B_n$ , and the edges between these vertices, and decide which vertices outside  $A_{\alpha-1, \beta+1}$  are red, and which of them are up-sites. For each vertex in  $A_{\alpha-1, \beta+1}$  having more than two 1-neighbours and/or having a down-site outside  $A_{\alpha-1, \beta+1}$  as a 1-neighbour, we decide if that vertex is red, and whether it is an up-site or a down-site (these vertices cannot be correctly configured). We then know which of the vertices outside  $A_{\alpha-1, \beta-2}$  are correctly configured and we decide which of them are open.

This leaves a set  $W$  of vertices in  $A_{\alpha-1,\beta+1}$  with at most two neighbours which are the ones that could be correctly configured. As at (3.5), the set  $W$  has at most  $12(\beta+2)^2$  elements and for  $x$  to have a chance of being 3-pivotal there must exist a subset  $W'$  of  $W$  such that if all the vertices in  $W'$  are open and all the vertices in  $W \setminus W'$  are closed then  $x$  is 3-pivotal. So if  $Y_i < p$  for all  $x_i$  in  $W'$  and  $Y_i > p$  for all  $x_i$  in  $W \setminus W'$ , then the event  $E'_{n,3}(x)$  occurs, where  $E'_{n,3}(x)$  denotes the event that  $x$  is 3-pivotal in a modified model where the diminishments are suppressed in  $A_{\alpha-1,\beta+1}(x)$ . Hence

$$P[E'_{n,3}(x)] \geq [p(1-p)]^{12(\beta+2)^2} P_{n,3}(x).$$

Adding or removing extra non-red vertices in  $A_{\alpha,\beta}(x)$  does not affect event  $E'_{n,3}(x)$  and therefore  $E'_{n,3}(x)$  is independent of the event  $R_n(x, \alpha, \beta)$ . Also, if  $E'_{n,3}(x) \cap R_n(x, \alpha, \beta)$  occurs, then there are at most  $12(\beta+2)^2$  correctly configured red vertices in  $A_{\alpha-1,\beta+1}$ , and the probability that none of these is diminished is at least  $(1-q)^{12(\beta+2)^2}$ . In this case event  $E_{n,3} \cap R'_n(x)$  occurs, and (6.4) follows with

$$\delta_1 := [p(1-p)(1-q)]^{12(\beta+2)^2} \exp(-\pi(\beta^2 - \alpha^2)\lambda(1-p)).$$

**Proof of Lemma 6.1.** Assume for now that  $30.5 < |x| < n - 30.5$ . Create the full process of intensity  $\lambda$  outside the circle  $C_{30}$  around  $x$ , and the red process  $\mathcal{P}_{p\lambda,29,30}$  in the annulus  $A_{29,30}(x)$ . Assuming there are no other vertices in  $A_{29,30}(x)$ , determine which vertices outside  $C_{30}$  are diminished, but do not yet diminish any vertices inside  $C_{30}$ . Deem open all vertices that are red and have not been diminished at this stage. Let  $V$  be the set of vertices now connected (by an open path) to  $B_{0,2}$  and let  $T$  be those vertices connected to  $\partial B_n$ . Let  $S$  be the remaining open vertices, and let  $\mathcal{E}$  be the edges on  $S$  inherited the original random connection model. Set  $\mathbf{S} = (S, \mathcal{E})$ .

Now create the red process  $\mathcal{P}_{p\lambda,20,29}$ . Let events  $H := H(V, T, \mathbf{S})$  and  $H' := H'(V, T, \mathbf{S})$  be as described just before Lemma 4.1 (with  $\mu = p\lambda$ ).

Event  $H$  must happen if  $E_{n,3}(x) \cap R'_n(x, 20, 29)$  is to occur. Hence by (6.4),  $P[H] \geq \delta_1 P_{n,3}(x)$ , and therefore by Lemma 4.1,  $P[H'] \geq \varepsilon(p\lambda)\delta_1 P_{n,3}(x)$ .

As in the latter part of the proof of Lemma 5.1, if  $H'$  occurs we can then form little discs  $D_1, \dots, D_{30}$  forming a path in  $C_{20}(x)$  from  $y^*$  to  $x$  and  $K_1, \dots, K_{30}$  forming a path in  $C_{20}(x)$  from  $z^*$  to  $x$ , this time with only  $D_{30}$  and  $K_{30}$  within unit distance of  $x$ . Then  $x$  will be 4-pivotal if we have exactly one red vertex in each of these discs, no other vertices in the rest of the process  $\mathcal{P}_\lambda \cap C_{30}$ , all edges along these paths are present,  $Y_0 < p$ ,

$x$  is a down-site but its neighbours are up-sites, and no vertices in  $C_{30}$  are diminished. This all occurs with probability at least

$$\delta_2(p, q) := (0.005^2 \pi \lambda p)^{60} [f(0.9)]^{62} \exp(-900 \pi \lambda) (p/8) (1 - q)^{12(31^2)},$$

where the last factor is a lower bound on the probability that no diminishment occurs in  $C_{30}$ , by the same argument as in the proof of (6.4). Therefore

$$P_{n,4}(x) \geq \delta_1 \delta_2 \varepsilon (p \lambda) P_{n,3}(x),$$

for  $x$  with  $30.5 < |x| < n - 30.5$ . For other  $x$  we can argue similarly to the last part of the proof of Lemma 3.2 to find some continuous  $\delta_3 : (0, 1)^2 \rightarrow (0, \infty)$  such that  $P_{n,4}(x) \geq \delta_3(p, q) P_{n,3}(x)$ . So taking  $\delta = \min(\delta_1 \delta_2 \varepsilon, \delta_3)$ , we are done.  $\square$

**Proof of Theorem 2.3.** We take  $q_0 < 1$ , and fix  $f$ . We now set  $q^* = (1 + q_0)/2$ , and choose  $\lambda > \lambda_{q_0 f}$ , and consider the graph  $RCM(\lambda, f)$ . Define  $p^* := \lambda_{q_0 f}/\lambda$ , so  $p^* \in (0, 1)$ . Now by considering a small box around  $(p^*, q^*)$ , and using Lemma 6.1, (6.1), (6.2) and the analogue of Proposition 3.1, we can find  $\varepsilon > 0$  such that  $\varepsilon < \min(p^*, 1 - p^*)$  and

$$\theta(p^* + \varepsilon, q_0) \leq \theta(p^*, q^*) \leq \theta(p^* - \varepsilon, 1).$$

Now the definition of  $p^*$  implies that  $RCM((p^* + \varepsilon)\lambda, q_0 f)$  percolates. Hence, on  $RCM(\lambda, f)$  the mixed site-bond process with parameters  $(p^* + \varepsilon, q_0)$  percolates and therefore the diminished site process with parameters  $(p^* + \varepsilon, q_0)$  percolates. Thus  $\theta(p^* + \varepsilon, q_0) > 0$  and hence  $\theta(p^* - \varepsilon, 1) > 0$  which means that  $RCM((p^* - \varepsilon)\lambda, f)$  percolates so

$$\lambda_f \leq (p^* - \varepsilon)\lambda < p^*\lambda = \lambda_{q_0 f}$$

and we are done.  $\square$

**Proof of Theorem 2.4.** Since  $S_p f \equiv S_{p/q} S_q f$ , it suffices to consider the case with  $q = 1$  so that  $S_q f \equiv f$ . Define the connection function  $g(r) = f(\sqrt{p}r)$ . Then  $S_p f \equiv pg$  so by Theorem 2.3,

$$\lambda_{S_p f} > \lambda_g$$

and hence the graph  $RCM(\lambda_{S_p f}, g)$  is the realization of a supercritical random connection model. Let  $p_c^{\text{bond}}$  and  $p_c^{\text{site}}$  denote the critical values for bond,

respectively site, percolation on this graph. By Theorem 2.2, we have  $p_c^{\text{bond}} > p_c^{\text{site}}$ .

Given  $p' \in (0, p)$ , we have  $p'g \equiv (p'/p)S_p f$  so that  $\lambda_{S_p f} < \lambda_{p'g}$  by Theorem 2.3, so that the graph  $\text{RCM}(\lambda_{S_p f}, p'g)$  does not percolate, and therefore  $p_c^{\text{bond}} \geq p'$ . Hence, taking  $p' \uparrow p$ , we have

$$p \leq p_c^{\text{bond}} < p_c^{\text{site}}. \quad (6.5)$$

By scaling, the graph  $\text{RCM}(\lambda_{S_p f}, g)$  is equivalent to the graph  $\text{RCM}(p^{-1}\lambda_{S_p f}, f)$  so that

$$p_c^{\text{site}} = \frac{p\lambda_f}{\lambda_{S_p f}}$$

and combining this with (6.5) yields the desired inequality  $\lambda_{S_p f} < \lambda_f$ .

For the last part, observe that  $\int_0^\infty r S_p f(r) dr = \int_0^\infty r f(r) dr$  and therefore the fact that (2.1) holds as a weak inequality for  $S_p f$  implies that it holds as a strict inequality for  $f$ .  $\square$

## References

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